UTRECHT UNIVERSITY

DOCTORAL THESIS

Inflation and Weyl symmetry in extended theories of gravity

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Inflation and Weyl symmetry in extended theories of gravity

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About the cover: The rivers that form from melting glaciers often do not follow an existing riverbed, instead the water is forced to find its own path, which results in the formation of streams of water of all scales. This is the reason why the structures one can see in this rivers exhibit scale invariance. This is also the subject studied in this thesis, although in the very different context of high energy physics, which is why this particular picture was choosen as the cover of this thesis.

Inflation and Weyl symmetry in extended theories of gravity

A new take on gauging Weyl symmetry

Inflatie en Weylsymmetrie in gemodificeerde theorieën van zwaartekracht

Een nieuwe kijk op het ijken van Weylsymmetrie

(met een samenvatting in het Nederlands)

Proefschrift

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door

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Abstract

Faculty of Science
Department of Physics and Astronomy

Doctor of Theoretical Physics

Inflation and Weyl symmetry in extended theories of gravity

by Stefano LUCAT

In this thesis we study both classical and quantum properties of a gauge realisation of Weyl symmetry, achieved by endowing the space-time manifold with a torsional connection. Some component of torsion can then act as a gauge connection, thus compensating the local transformations of the metric tensor and the fields.

This leads to a rather generic physical theory, described in chapter 2, which features a dilaton and a vector field on top of the usual particle content of the standard model. We then study a possible mechanism that spontaneous breaks conformal symmetry, showing that cosmological inflation can be described in such a framework. The resulting model has notable features and agrees with inflationary observables in a wide range of the parameter space.

In chapter 4 we discuss detection of torsion by means of gravitational waves detectors, which is within range of experimental approach once the space based gravitational detectors will be online. Finally in chapter 5 we address the conformal anomaly and present first evidence that the extended Ward identity, which take into account the additional vector field, hold in a perturbative dimensional renormalization scheme, even when anomalous contributions are generated in the energy momentum tensor trace.

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Dedicated to my parents, Piero & Manuela,

Chapter 1

Introduction

The main subject of this thesis is the study of Weyl symmetry, in extended theories of gravity coupled to scalar degrees of freedom. This is a symmetry principle that assumes the only physical quantities are dimensionless ratios, and no absolute concept of scale exists in nature. Throughout this work we will use Weyl symmetry as a guiding principle, following its consequences in a range of applications: from early universe cosmology, to gravitational waves observations, to the quantum anomalies that emerge in these theories. In this introduction, I will explain the concepts elaborated in the thesis in simpler words, to allow the non expert audience to have a feeling for what will be discussed later.

1.1 Making the case for conformal symmetry

Any physics student knows that physical experiments are a measurement of dimensionless ratios, which is then converted in physical units (*e.g* meters, seconds, Joules, $\cdots$) by relating such ratio to a reference scale. If a different reference scale is used to perform the experiment, we do not expect the result to change. This simple concept, however, does not find a straightforward application in high energy physics and in gravitational theories, because of the existence of a fundamental dimensionfull coupling constant, namely the gravitational constant, G_N (Newton constant), which sets the strength of the gravitational interaction.

The Newton constant, when combined with the speed of light c and the Planck constant $\hbar$, produces a length scale, the Planck length,

$$L_P^2 = \frac{\hbar G_N}{c^3}\,, \tag{1.1}$$

which is thought to be fundamental in many theories, such as string theory or quantum loop gravity. In these theories it represents the length at which one expects the universe to start looking very different than at larger scales. For example, one could interpret the Planck length (1.1) as being the lattice spacing on a discretized space-time (as in causal set theory, dynamical triangulation, space-time foam), or being the average size of a quantum string.

There are valid points in both these approaches, however the existence of the Planck length (1.1) is incompatible with a scale invariant, and in particular a locally scale invariant universe. A way of picturing this, is to imagine scale invariance as the symmetry of fractal-like structure. It does not change as we zoom in or out, manifesting self-similar appearence no matter what scale we use to probe it. On the other side we have a crystal like structure, equipped with a fundamental length scale, the lattice spacing of its fundamental cell.

Many phenomena in nature exhibit self similarity, such as biological structures, systems near a first order phase transition, quantum theories at critical point of the renormalization flow, turbolent flow. Of course, many physical systems also present discrete structures, and the idea that a somewhat minimal length exists is certainly appealing.

This is not the approach considered in this thesis. By contrast, we try to construct a theory of gravity and matter where no intrinsic notion of scale, such as the one in (1.1), exists. In such a theory, all physical length scale would be spontaneously generated by radiative corrections, as in the Coleman-Weinberg mechanism [1]. In this picture, the radiative corrections to the effective potential generate a minimum which does not correspond to the symmetric state, which leads to a spontaneous condensation and the generation of a length scale.

This is motivated by empirical and theoretical arguments, some of which we report as follows.

The empirical motivations come from cosmology, the study of the universe on the largest scales ($> 100Mpsc$ [1]).

Under the assumption of homogeneity and isotropy, the metric field describing the universe can be written as,

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}^2\,, \tag{1.2}$$

where c is the speed of light, $\mathrm{d}\vec{x}$ measures comoving distances, which are kept time independent, and $a(t)\mathrm{d}\vec{x}$ is the physical infinitesimal distance. $a(t)$ describes therefore how the physical distances change as a function of time.

The perturbations from homogeneity are what we think seeds the original structures of the universe, and later develop into galaxies and the large scale structures we observe today. We characterise the perturbations by their statistical properties, which are encoded in the 2-points correlator of density fluctuations, whose Fourier transform defines the power spectrum $\mathcal{P}(k)$ as,

$$\langle\delta\rho(t,\vec{x})\delta\rho(t',\vec{x}')\rangle = \frac{1}{(4\pi G_N a(t)a(t'))^2}\int \mathrm{d}\vec{k}k^4\mathcal{P}(k)e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\,. \tag{1.3}$$

$P(k)$ is directly related to the perturbation of the gravitational potential, Φ, through the Poisson equation,

$$\vec{\nabla}^2\delta\Phi = \frac{4\pi G_N}{a^2(t)}\delta\rho\,,$$

therefore its form determines the initial gravitational perturbation that creates structures. Early studies on galaxies formation [2, 3], much before the

[1] $1Mpsc \simeq 3.08 \times 10^{22}m$.

observations of the cosmic microwaves background by WMAP, established a remarkable fact: that the spectrum of density perturbations is flat, independent on the scale (or time) at which the perturbation was generated. This was later confirmed by the observation of the cosmic microwaves background, the relic radiation poetically named the echo from the Big Bang, which revealed a slightly tilted spectrum, with the deviations from exact scale invariance being on the order of few percent.

Inflation (see chapter 4) provides an explanation for this fact, by linking the scale invariance to the approximate scale symmetry of quasi-deSitter space-times. However, in the context of our discussion, we might invoke a different explanation, that the scale symmetry is inherited from the Weyl invariance of the theory of Nature at very high energies.

In models possessing Weyl symmetry, one in fact obtains quite generically a scale invariant spectrum [4], because of the absence of any relevant length scale which would spoil this property. Furthermore, one can argue that, if Weyl symmetry is restored in the high energy regime of Nature, the quantum state of the fields near the Big Bang time should respect it as well. This means that, in fact, a scale invariant spectrum of perturbations is a rather natural consequence of Weyl symmetry.

A well known fact is that the universe is currently expanding, or that the scale factor $a(t)$ in (1.2) is increasing in time, as was showed for the first time by the Hubble law [5], and confirmed by more precise experiments afterwards (*e.g.* [6, 7]). Most of the observations that demonstrate this principle are based on measuring the redshift of photons. The observed frequency is related to the frequency at emission by,

$$\nu_{obs} = \frac{a_{em}}{a_{obs}} \nu_{em} \,,$$

where $a_{em} = a(t_{em})$ ($a_{obs} = a(t_{obs})$) denotes the cosmological scale factor at the emitter (observer). Knowing what the emission frequency should be, by looking at sources whose spectrum is known, allows us to tell whether the observed frequency is bigger, or smaller, than what it should be, *i.e.* whether $a(t)$ is increasing or decreasing. However, as noticed in [8, 9], there could be another interpretation: that the masses of particles are increasing as time goes by. If the mass of the atom which emitted the photon was smaller, the Bohr radius of the atom which emitted it was also smaller, and therefore the emitted frequency will be measured as if the universe was expanding [8]. In this interpretation, the space-time metric can be static, but the masses of particles change, and therefore one would observe redshift in frequency.

In the above example, we encountered an example of Weyl transformation: measurements of the gravitational field are found to be the same if a local rescaling of dimensionfull quantities (in the example masses) is performed. This is so because measurements can only probe dimensionless ratios, in this case length over proton mass. Choosing which one is constant and which one varies are interpretations of the measurement, and should be regarded as a gauge transformation, as coordinate transformations are.

It can be argued that these two interpretations are not really different [9, 10], but are related by a frame transformation. In mathematical terms, such a transformation can be written as,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \to \Omega^2(x)\mathrm{d}s^2\,, x^\mu \to x^\mu\,. \tag{1.4}$$

It is worth pausing for a moment to clear a semantic issue, namely the distinction between Weyl and conformal symmetry. A conformal transformation, in fact, is a coordinate transformation, $x^\mu \to x^\mu + \xi^\mu$, with the property that

$$\mathcal{L}_\xi g_{\mu\nu} \equiv \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\nu\alpha}\partial_\mu \xi^\alpha = \Omega^2(x) g_{\mu\nu}\,, \tag{1.5}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the direction ξ^μ, and describes how the metric change under a infinitesimal coordinate transformation along the direction ξ. The solutions to equation (1.5), $\xi_i = \xi_i^\mu(x)\partial_\mu$, define the conformal group associated with the manifold $(\mathcal{M}, g)$, via the local Lie algebra structure,

$$[\xi_i\,,\xi_j]^\nu \partial_\nu \equiv \left(\xi_i^\mu(x)\partial_\mu \xi_j^\nu(x) - \xi_j^\mu(x)\partial_\mu \xi_i^\nu(x)\right)\partial_\nu\,.$$

The conformal group is then the Lie group that is associated from this algebra. In flat space-time, $g_{\mu\nu} = \eta_{\mu\nu}$ such a Lie group is $SO(2,4)$, which has 15 generators, the usual generators of conformal transformations, corresponding to the 15 linearly indipendent solutions of (1.5) in flat space-time. Therefore, this symmetry group is a global group, in the sense that its generic element can be expanded in the 15 generators of the group.

In the flat space case, the solutions of (1.5) define $\Omega(x) = \lambda(1 - 2b^\mu x_\mu + b^2 x^2)$, where λ is a constant, and b^μ an arbitrary direction in flat space. This makes clear the global nature of conformal symmetry, since it is a group with finitely many generators. By contrast, the form factor of a Weyl transformation, as in (1.4), is not limited to a quadratic polynomial in the space-time coordinates, but is arbitrary up to singular rescalings. In this sense we call a Weyl transformation a gauge symmetry: because it has infinitely many generators!

In the previous example we argued that different scales might be associated to different moments of the cosmic evolution. Since time and space are on equal footing in gravitational theories, we should expect variations of the energy scale also in different space regions. From this, we justify the local nature of (1.4): the scale used by local observers to perform experiments can vary locally, and be different in different regions of space-time. If we follow this interpretation, we are lead to consider the *local scale transformation* (1.4) as a symmetry of Nature, which is broken today, but realised in the fundamental theory.

There are two pieces of theoretical evidence that support this conclusion: firstly, the fact that the Standard Model is a nearly conformal theory. Except for the Higgs mass (and the neutrino's which could in fact be Yukawa generated, as the electron is), all the interactions of the Standard Model possess dimensionless couplings (in 4 dimensions) and safe for the scalar mass and kinetic terms, all other terms of the Standard Model action are classically conformal.

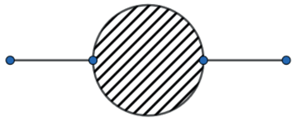


FIGURE 1.1: The Feynman diagram contributing to the Higgs particle mass. Here y_f are the Yukawa couplings of fermions to the Higgs particle, $y_f \bar{\psi} H \psi$.

In addition, it has been argued [11, 12] that Weyl symmetry can be used to solve the gauge hierarchy problem. In computing loops corrections to the Higgs mass, one finds (using a cutoff regularization scheme), $\delta m_H \propto \Lambda^2 y_f^2$, where y_f is the Yukawa coupling of the Higgs to some scalar and Λ the cutoff scale. If one postulates that the standard model describes physics up to the Planck scale, *i.e.* $\Lambda \simeq M_P$, he or she would conclude that the natural scale for the Higgs boson's mass is the Planck mass. This is clearly in contrast with experiments, since $m_H = 125GeV$ and a huge amount of fine tuning is required to keep under control the quantum corrections to it.

Supersymmetry provides an elegant explanation for the gauge hierarchy problem, by introducing supersymmetric partners in such a way that the bosonic corrections to the Higgs mass, always cancel against the fermionic corrections, up to logarithmic contributions. However, since supersymmetry is disfavoured by particle accelerator observations, we need a different explanation, which can be found by invoking Weyl symmetry.

Since Weyl symmetric theories forbids the existence of fundamental length scales, it also forbits the introduction of cutoff scale Λ. Therefore, it selects dimensional regularization as the favorite procedure to define the quantum effective action [11]. In this scheme, one extends the Lorentz symmetry $SO(1,3)$ to the more general $SO(1, D-1)$, and computes loop integrals in accordance to this generalised symmetry, *i.e.* by the prescription $\mathrm{d}^4k \to \mathrm{d}^D k$ and the extension of the momentum k^μ and Lorentz covariant tensors to D dimensions. The physical answer is obtained by analytically extending the result of this procedure to the regions of the complex D plane where it was not defined, namely to $D = 4$.

Dimensional regularization also respects Lorentz and Gauge invariance, and only minimally breaks scale symmetry through logarithmic corrections. In [11] the authors use this to show that minimal conformal extensions of the standard model are in fact able to solve the hierarchy problem. They extend the standard model by adding a scalar particle, φ, charged under some (hidden) gauge group, which condenses and dynamically generates a mass for the Higgs field. The physical Higgs particle will then be a mixture of φ and the standard model Higgs, the mixture with lowest mass eigenstate.

The second motivation comes from the study of the renormalization - group flow and of quantum theories near critical points. These are points where the correlation length goes to infinity, as for example happens in the Ising model near the critical temperature, which implies that the physical behaviour of the system becomes self similar.

In the language of quantum theories, we understand criticality through renormalization: loop corrections introduce a scale dependence in the coupling constants of the theory, in cut off renormalization through the highest momentum that goes in the loop. This scale dependence can be used to resum certain diagramatic contributions, through the renormalization group equations.

It is a well known fact that the renormalization group's equations can possess fixed points [13]. These are energy scales at which the coupling constants of the theory's cease to be scale dependent. At this point, the beta functions of the theory, describing how the coupling constants change as we rescale lengths, vanish and criticality is realised at the quantum level. This then enhances the symmetry of the quantum field theory to the full conformal group, $SO(2,4)$. Arguably, the best analytical understanding of strongly interacting quantum systems, at the critical points, comes from the enhanced conformal symmetry realised near the critical points. A stunning variety of systems can be understood through this approach, from magnets, to the asymptotic freedom of the strong interaction of quarks.

If we conjecture that the fundamental theory of nature should possess renormalizability, we have to consider the existence of an ultra violet fixed point of the renormalization group equations. Precisely at this point, the rescaling (1.4) becomes an enhanced symmetry of the quantum theory, and the powerful techniques of conformal field theory used to study its property. This is the aim of the asymptotic safety for quantum gravity conjecture [14, 15]. The problem with this approach, however, is that it cannot resolve the unitarity problem in quantum gravity, first pointed out in [16].

The missing piece in this whole argument, however, is gravity. If we want to argue that conformal transformations are on the same footing as coordinate transformations, *i.e.* they merely describe different observers, we should be able to construct a theory of gravity for which the metric rescaling (1.4) is a symmetry. In such a theory, all gravitational observables must be the same in all conformal frames. At the present state of art, there is no theory that is capable of this: although conformal lagrangians can be constructed using the Weyl tensor or a non minimally coupled scalar [17, 18], observables such as curvature and geodesics trajectories are frame dependent in these theories, and we are therefore forced to strongly modify Einstein's theory of gravity and explain how to flow from the high energy description to the low energy limit (Einstein's theory). Furthermore, the gravitational models based on Weyl tensor are generally unstable, both at the classical and quantum level, as they violate Ostrogradsky's theorem [19] and possess ghosts [20].

The approach advocated here is that, in order to achieve invariance under the Weyl transformation (1.4) it is necessary to introduce an additional field, which will act much like an abelian gauge connection for Weyl transformations, and study its physical properties, both at the classical and quantum

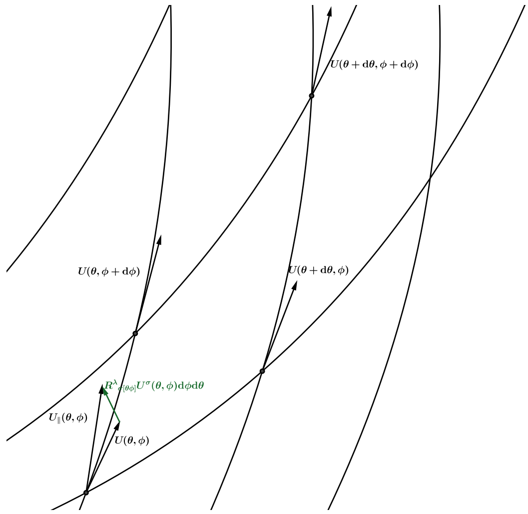


FIGURE 1.2: An illustration of parallel transport on a 2 dimensional surface. The vector U is parallel transported along a closed path, and ends in a different configuration from what it started. The curvature tensor quantifies by how much $U_{\parallel}$ is displaced from U.

level.

1.2 Gravity with torsion and its relation to Weyl symmetry

Einstein's theory of general relativity is a geometric theory, which couples the geometrical notion of curvature with the energy momentum tensor of the matter fields. This is not different in the extended theory of gravity we will study in this thesis, with the difference that additional geometrical notions are required to accomodate Weyl symmetry. To this end, consider a manifold $\mathcal{M}$ equipped with a metric field $g_{\mu\nu}$ that is used to measure distances, and an affine connection, ∇_{μ} that defines parallel tranport, according to

$$V^{\mu}_{\parallel}(x + \mathrm{d}x) = V^{\mu}(x) + \nabla_{\lambda} V^{\mu}(x) \mathrm{d}x^{\lambda} \,. \tag{1.6}$$

When the covariant derivative acts on a basis vector $\hat{e}^{(\mu)}$, it defines the manifold connection, which can be written as,

$$\nabla_{\mu} \hat{e}^{(\lambda)} = \Gamma^{\lambda}{}_{\sigma\mu} \hat{e}^{(\sigma)} \,, \nabla_{\mu} V^{\lambda} = \partial_{\mu} V^{\lambda} + \Gamma^{\lambda}{}_{\sigma\mu} V^{\sigma} \,. \tag{1.7}$$

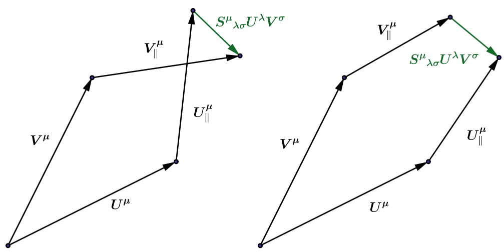


FIGURE 1.3: The geometrical meaning of torsion in the parallelogram construction. We slide two vectors along each other in a self parallel manner, after which they have turned and do not touch each other any more. Torsion measures by how much they fail to do so.

In figure 1.2 we see an illustration of how parallel transport works on a curved surface. The curvature tensor quantifies by how much a vector changes during this procedure, and is defined by,

$$\left(\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}\right) V^{\lambda} = R^{\lambda}{}_{\sigma\mu\nu} V^{\sigma}\,. \tag{1.8}$$

In physical terms, parallel transport defines what it means to move 'straight', and in particular what is a straight line, on a curved manifold. This is achieved through the geodesic equation, which is the curved defined by,

$$\dot{x}^{\mu}\nabla_{\mu}\dot{x}^{\lambda} = 0\,,\ \dot{x}^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}\,, \tag{1.9}$$

and λ is the affine parameter we use to parametrize the curve, $x^{\mu}(\lambda)$. The concept of geodesics is used to generalize the second law of Newton to gravitational systems: under no additional force than gravity a point-like body will move along geodesics of the space-time manifold. The proper length of the geodesic, measured by integrating,

$$\tau = \int \sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}\mathrm{d}\lambda\,,$$

defines the notion of proper time, which has the physical meaning of subjective time experienced by point-like observers. If the geodesic is parametrized by proper time, $\lambda = \alpha\tau + \beta$ with $\alpha\,, \beta$ constants, we then have $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \frac{1}{\alpha^2}$.

In extended theories of gravity, contrary to what we learn in books about general relativity, it is not the case that a geodesic is a curve of minimal length. This is because the connection $\Gamma^{\lambda}{}_{\sigma\mu}$ is allowed to be arbitrary. The additional geometric notions that this allows are that of torsion and non-metricity. The first can be thought as a twisting of space-time, which is illustrated in figure 1.3–1.4. To understand the figure, consider that the torsion tensor,

$$T^{\lambda}{}_{\mu\nu} \equiv \Gamma^{\lambda}{}_{[\alpha\beta]}\,, \tag{1.10}$$

picks up the anti-symmetric part of the manifold connection. Then parallel transporting a vector V on the direction U and viceversa, yields to,

$$V^{\mu}_{\parallel}(x+U) - U^{\mu}_{\parallel}(x+V) = S^{\mu}{}_{\alpha\beta} V^{\alpha} U^{\beta} \,,$$

which is precisely what is drawn in figure 1.3.

The meaning of non-metricity can be understood as the symmetrised version of the connection,

$$\nabla_{\mu} g_{\alpha\beta} = \partial_{\mu} g_{\alpha\beta} - 2\Gamma_{(\alpha\beta)\mu} \neq 0 \,.$$

For example, Weyl gravity This would signify that the metric is not parallel transported on the manifold $\mathcal{M}$. For two observers moving through paths $\gamma_1\,,\gamma_2$ would measure a discrepancy in the metric, given at linear order by,

$$g^{(1)}_{\mu\nu} - g^{(2)}_{\mu\nu} = \int_{\Sigma[\gamma_1,\gamma_2]} \left(\partial_{\sigma}\Gamma_{(\mu\nu)\lambda} - \partial_{\lambda}\Gamma_{(\mu\nu)\sigma}\right) \mathrm{d}x^{\lambda}\mathrm{d}x^{\sigma} \,, \tag{1.11}$$

where $\Sigma[\gamma_1,\gamma_2]$ denotes the surface delimited by $\gamma_1 \cup \gamma_2$ and we employed stokes theorem.

The first person to consider the possibility of introducing a gauge connection to compensate for the transformation (1.4) was Hermann Weyl, who proposed to use a non-metricity vector (Weyl vector) for such a role. This seems a very natural way of realising the concept, as can be seen from the definition of Weyl vector W_{μ},

$$\nabla_{\mu} g_{\alpha\beta} = W_{\mu} g_{\alpha\beta} \,, \tag{1.12}$$

where ∇ is the covariant derivative that generates parallel transport on the space-time manifold.

However, soon after Weyl proposed his theory, Einstein replied with a criticism: since the metric is used to compute the proper time, which in general relativity provides the only absolute notion of time and distance, it cannot change under parallel transport. If that was the case, different observers taking different paths and meeting again would have no way of agreeing on how much time has passed, or on how to measure distances. This is because of the presence of the Weyl vector (1.12). Indeed, evaluating (1.11) for the case (1.12) we find that,

$$g^{(1)}_{\mu\nu} - g^{(2)}_{\mu\nu} = \int_{\Sigma[\gamma_1,\gamma_2]} g_{\mu\nu} \left(\partial_{\sigma} W_{\lambda} - \partial_{\lambda} W_{\sigma}\right) \mathrm{d}x^{\lambda}\mathrm{d}x^{\sigma} \,, \tag{1.13}$$

which is thus dependent on the flux of the transverse part of the Weyl field through the surface $\Sigma[\gamma_1,\gamma_2]$.

This argument dissuaded Weyl, and many after him, to pursue the theory. However, there is a different, more subtle way of achieving the same result, using a space-time torsion vector instead of non-metricity. In this theory, the only assumption from General Relativity that is dropped is that of a symmetric connection.

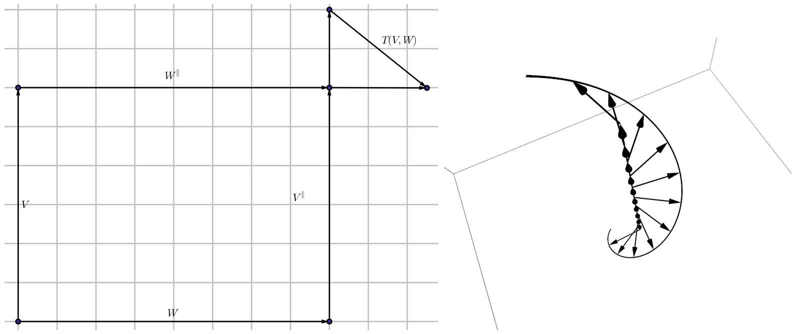


FIGURE 1.4: On the left, the effect of the torsion trace on parallel transport of vectors: the torsion trace induces a rescaling of vectors during parallel transport. Note that the parallel transported version of V and W are parallel respectively to V and W, and lie on the same plane as their parent vectors. This feature is characteristic of the torsion trace and does not hold when more components are added to the torsion tensor. On the right, an example of a space with torsion, and the rotation induced by parallel transport. In this space, geodesics are helicoidal curves, winding around the z axis, or vertical lines in the z direction. The vectors pointing in horizontal directions are here the Jacobi field for these two types of geodesics.

To build an intuitive understanding of why this might be true, we go back to the geometrical meaning of torsion, illustrated in figure 1.3. Since torsion is a 3 indices tensor, antisymmetric in two of them, it can be decomposed, according to the Young classification, in 3 irreducible components. These would be one trace, $\mathcal{T}_\mu = \frac{2}{D-1} S^\lambda{}_{\mu\lambda}$, a totally antisymmetric part, $\Sigma_{\alpha\mu\nu} = g_{\lambda[\alpha} S^\lambda{}_{\mu\nu]}$ and a third component, which has mixed symmetries.

The totally antisymmetric part would induce, during parallel transport of U along V, a rotation of the vector U on the plane orthogonal to V. For this reason, $\Sigma_{\alpha\mu\nu}$ is linked to parity and chiral transformations, but not to conformal symmetry. Analogously, the third irreducible component would induce rotation and shear of vectors, but since it is traceless, no rescaling.

The geometrical role of the torsion trace, $\mathcal{T}_\mu$, is such that the parallelogram construction in figure 1.3 would lie in the plane spanned by V and W. This is a complicated way of describing a rescaling of the vectors, which is the geometrical reason why $\mathcal{T}_\mu$ is associated with Weyl symmetry.

In other words, the local rescaling of the metric (1.4) does not map the manifold into itself, but generates terms which happen to be the same as if a torsion trace $\propto \partial_\mu \log \Omega$ was present. As we explain in chapter 2, this property can be used to construct a gauge theory, since any local rescaling of the type (1.4) can be compensated by the appropriate transformation of the torsion trace.

We have not yet explained how this realization of Weyl invariance evades Einstein's criticism to Hermann Weyl proposal. The explanation is slightly technical, and roots in the fact that non-metricity and torsion are two independent geometrical concept, and one can construct geometries with one or the other, or both. A theory with torsion but without non-metricity has the property that,

$$\Gamma_{\alpha[\mu\nu]} = S_{\alpha\mu\nu}\,, \Gamma_{(\alpha\beta)\mu} = \frac{1}{2}\partial_\mu g_{\alpha\beta}\,. \tag{1.14}$$

A solution for the system (1.14) exist, and represents a geometry having a parallel transported metric and non vanishing torsion.

In chapter 2 we will see how this can be used to construct a Weyl invariant theory of gravity. The result will be more complicated than the original proposal by Weyl, but works in a similar way.

1.3 Weyl anomaly and massless degrees of freedom

So far we have heuristically argued in favour of Weyl symmetry, but have not presented evidence that such a framework is actually needed from a theoretical standpoint. In other words, why would we want to introduce new degrees of freedom in the fashion of torsion, other than for pure mathematical beauty? The answer to this question stands in the problem known as the conformal anomaly, or trace anomaly. Before dwelling into this discussion, though, we have to review the concepts of Ward-Takahashi identities and the beautiful mathematical framework surrounding quantum anomalies.

Ward-Takahashi identities are simply quantum mechanical analogues of the well known second theorem by Emmy Noether, which states that to any symmetry generator that leaves the lagrangian invariant, there corresponds a conserved charge. This theorem holds in any classical theory, and the symmetries that are mentioned are global symmetries, parametrized by a set of constant parameters.

In quantum field theory, we deal with gauge symmetries, which are local as they depend on an free function that can be chosen arbitrarily. Ward-Takahashi identities are constrain's on the quantum correlators of fields that the gauge symmetry imposes.

A typical example, is the case of quantum electrodynamics, a theory defined by the Lagrangian,

$$\mathcal{L} = -\frac{i}{2}\left(\bar{\psi}\gamma^\mu\left(\partial_\mu - ieA_\mu\right)\psi - \bar{\psi}\gamma^\mu\left(\partial_\mu + ieA_\mu\right)\psi\right) + m\bar{\psi}\psi\,, \tag{1.15}$$

which has the conserved charge,

$$Q = e\int \mathrm{d}\vec{x}\,\langle\bar{\psi}\gamma^0\psi\rangle = -e\int \mathrm{d}\vec{k}\,\langle\hat{N}_F(\vec{k}) - \hat{N}_{\bar{F}}(\vec{k})\rangle\,,$$

where $\hat{N}_F(\vec{k})$ (and $\hat{N}_{\bar{F}}(\vec{k})$) are the number densities operators, that count the number of particles (antiparticles). Indeed, since particles have negative charges and antiparticles positive charges, Q counts the electric charge contained in the quantum state of the field.

The Ward-Takahashi identity associated with the theory (1.15) would be,

$$\partial_\mu \langle J^\mu \rangle \Big|_A = 0 \, , J^\mu = e \, \bar{\psi} \gamma^\mu \psi \, . \tag{1.16}$$

In (1.16) the brackets $\langle \cdot \rangle \Big|_A$ signify that the expectation value is to be taken in the presence of an arbitrary electromagnetic field A_μ. Therefore the identity (1.16) is to be taken as a functional relation, and is usually evaluated by taking functional derivatives with respect to A_μ and then setting it to zero. This allows to evaluate the coefficients in the expansion,

$$\begin{aligned} \langle J^\mu(x) \rangle =& J_0^\mu(x) + \int \mathrm{d}^D y \, \Pi^{\mu\alpha}(x,y) A_\alpha(y) \\ &+ \int \mathrm{d}^D y \mathrm{d}^D z \, \Gamma_{(2)}^{\mu\alpha\beta}(x,y,z) A_\alpha(y) A_\beta(z) + \cdots \, , \end{aligned} \tag{1.17}$$

where J_0^μ is the tree level result and the coefficients,

$$\Gamma_{(n)}^{\mu\alpha_1 \cdot \alpha_n} = \frac{\delta^{(n)} \langle J^\mu \rangle}{\delta A_{\alpha_1} \cdots \delta A_{\alpha_n}} \Bigg|_{A=0} \, ,$$

have the interpretation of proper vertices. For example, $\Pi^{\mu\alpha}(x,y)$ has the physical meaning of the *vacuum polarization* of the photon. From (1.17) we can see how this represents the response, of the electron field, to a small perturbation in the electromagnetic field. $\Pi^{\mu\alpha}(x,y)$ measures the linear response, while higher n's $\Gamma_{(1)}^{\mu \cdots}$ capture the more subtle effects of the interaction, such as particle production.

When functional derivatives act on (1.16) the yield novel identities for the coefficients $\Gamma_{(n)}^{\mu\alpha_1 \cdot \alpha_n}$, in the case (1.15) the momentum space relations,

$$k_\mu \Gamma_{(n)}^{\mu\alpha_1 \cdot \alpha_n}(p^1, \cdots, p^n) = 0 \, , k_\mu = \sum_{i=1}^{n} p_\mu^i \, .$$

As one can read in any quantum field theory text book, for example in [21], this identities imply that the photon vacuum polarization remains transverse, and that no quantum correction to its longitudinal polarization are generated, to any order in perturbation theory. This implies that the electromagnetic field remains massless, to all orders in perturbation theory.

As one can imagine the Ward-Takahashi identities are very powerful, as they allow to perform powerful consistency checks of perturbative calculations. However, one has to be careful when handling them, because of the presence of quantum anomalies. Mathematically, anomalies are a result of renormalizing the quantum field theory, as they appear when one consistently renormalizes loop diagrams, such as the one we see in figure 1.5.

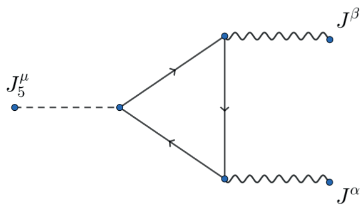


FIGURE 1.5: The feynman diagram contributing to the chiral anomaly of QED or QCD. It represents the three points function with one chiral current, $J_5^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$ and two $J^\alpha = \bar{\psi}\gamma^\mu\psi$.

To illustrate this example, consider again the action (1.15): other than being invariant under the transformation, $\psi \to e^{i\alpha}\psi$, we have the symmetry, $\psi \to e^{i\gamma^5\alpha}\psi$, which holds in the massless limit,$m \to 0$, and yields to the conservation law,

$$\partial_\mu \langle \bar{\psi}\gamma^5\gamma^\mu\psi \rangle = 0\,,$$

which in turns implies the conservation of the chiral charge,

$$Q_5 = e\int \mathrm{d}\vec{x} \langle \bar{\psi}\gamma^5\gamma^\mu\psi \rangle = e\int \frac{\mathrm{d}^3k}{(2\pi)^3} \langle \hat{N}_R(\vec{k}) - \hat{N}_L(\vec{k}) \rangle\,, \tag{1.18}$$

where $\hat{N}_L(\vec{k})$ (and $\hat{N}_R(\vec{k})$) are the number densities operators that count the number of particles of left and right chirality (obtained by projecting $\frac{1\pm\gamma^5}{2}\psi$).

However, the contribution from the diagram in figure 1.5 yields to the result,

$$\begin{aligned} \partial_\mu \langle \bar{\psi}\gamma^5\gamma^\mu\psi \rangle &= \frac{e^2}{32\pi^2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}\,, F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \\ \implies \dot{Q}_5 &= \frac{e^2}{32\pi^2}\int \mathrm{d}\vec{x} F_{0i}F_{jk}\epsilon^{ijk} = \int \mathrm{d}\vec{x}\vec{E}\cdot\vec{B}\,, \end{aligned} \tag{1.19}$$

where $\vec{E}$, $\vec{B}$ are the electric and magnetic fields respectively. The physical meaning of the result (1.19) can be understood as the production of Dirac's particle pairs induced by an external electromagnetic field.

In [22] the author exhamines fermions on a line (a 2 dimensional system) with antiperiodic boundary conditions. In figure 1.6 we see a picture of the spectrum $E(p)$, of such a system: the two branches correspond to massless dirac fermions, in 2 dimensions, and the sign of the relation $E = \pm p$ determines also the chirality of the state. The allowed states are discretized by the periodic boundary conditions, and the $p = 0$ state forbitten by the antiperiodicity.

The effect of applying an electromagnetic field to the system is that particles get shifted to the right or to the left of the diagram, depending on the external field, creating a hole in the Dirac sea. This hole can now be filled

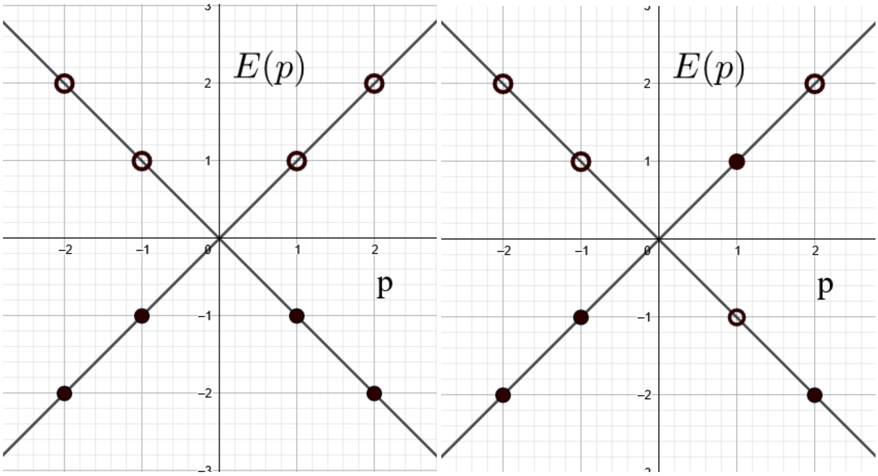


FIGURE 1.6: The manifestation of the (massless) QED anomaly for a system with discrete spectrum. On the left hand side, we see the vacuum state of the theory: the filled dots represent the filled Dirac sea, while the white dots are the allowed excitations, corresponding to physical particles. On the right we observe the effects of applying a external electromagnetic field: the Dirac sea with a certain chirality get shifted upwards, while the one with opposite chirality downwards.

by one particle in the sea below, which pushes the hole deeper, until it eventually disappears since the repository of negative energy states is infinitely deep. The same happens on the opposite chirality branch, which eventually leads to the situation on the right hand side of figure 1.6.

Since the $E = +p$ branch has positive chirality, and the $E = -p$ branch has negative chirality, the chiral charge of the system is not conserved, and the anomaly (1.19) precisely measures the rate at which such a violation occurs per unit volume, as can be inferred by Eq. (1.19) and dimensional analysis.

This process is described by the $2D$ anomaly equation, namely,

$$\begin{aligned} \partial_\mu \langle \bar{\psi}\gamma^5\gamma^\mu\psi\rangle =& \frac{e}{2\pi}\epsilon^{\alpha\beta}F_{\alpha\beta}\,, F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \\ \implies \dot{Q}_5 =& \frac{e}{\pi}\int \mathrm{d}x F_{0x}\,, \end{aligned} \tag{1.20}$$

which shows that the anomaly represents a rate, the rate at which the right-left chirality particle pairs are created. In the language of figure 1.6, $\dot{Q}_5$ has the dimension of an inverse time, the time that it takes, on average, for the process described in figure 1.6 to happen. Integrating the last line of (1.20) in

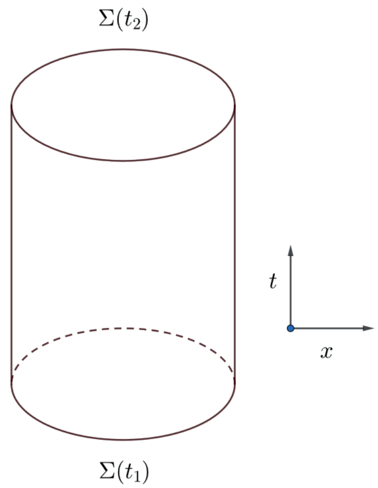


FIGURE 1.7: The surface of integration in Eq.(1.21) in $2D$ QED. If the winding number of the electric potential, θ in (1.21), changes between the two surfaces, particle's pairs as in figure 1.6 are created.

time, between two times $t_1\,, t_2$, we find,

$$\int_{t_1}^{t_2} \mathrm{d}t\, \dot{Q}_5 = Q_5(t_2) - Q_5(t_1) = \frac{e}{\pi}\int \mathrm{d}x\mathrm{d}t F_{0x} = \frac{e}{\pi}\left(\int_{t_2} \mathrm{d}x A_x - \int_{t_1} \mathrm{d}x A_x\right)$$
$$= \frac{1}{\pi}\left(\int_{t_2} \mathrm{d}x\partial_x\theta - \int_{t_2} \mathrm{d}x\partial_x\theta\right) = \frac{1}{\pi}\left(\int_{t_2} \mathrm{d}\theta - \int_{t_1} \mathrm{d}\theta\right) = 2(n_2 - n_1)\,, n \in \mathbb{N}\,, \tag{1.21}$$

since in $2D$ the the field strength $F_{\alpha\beta}$ has only one component, which implies $A_x = \frac{1}{e}\partial_x\theta$, and the contributions from the $x = const$ part of the integral vanish (see figure 1.7). Finally, making use of the periodic bounday conditions on constant t slices, we have that the integral in $\mathrm{d}\theta$ is quantised, and has to be a integer multiple of 2π. The right hand side of (1.21) measure the change in winding number of θ around the spacial sections.

Notice that, since a gauge transformation changes $\theta \to \theta + \alpha$, if the gauge transformation has non zero winding number, in the sense of (1.21), it can change the charge Q_5. This class of gauge transformations, usually referred to as 'large gauge transformations', has a physical meaning, namely, as can be seen in (1.21), it leads to the production of particle's pairs. This example illustrates the topological nature of the chiral anomaly: since the relation between the gauge group, $U(1)$, is the same as the boundary in the example above, the vacuum state of the theory has to be characterised by the winding number of the gauge group on such boundary. If the winding number change, we have a vacuum transition and the production of a particle pair.

A similar property can be observed in $4D$, in the case of non abelian symmetry group, such as QCD or the weak interaction, we have $\mathcal{F}^{a}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta + ig\left[\mathcal{A}_\alpha\,,\mathcal{A}_\beta\right]$, where g is the gauge theory coupling, which implies that,

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t\, Q_5 = \int_V \mathrm{d}^4x\, \epsilon^{\alpha\beta\gamma\delta}\mathrm{Tr}\mathcal{F}_{\alpha\beta}\mathcal{F}_{\gamma\delta} = \\ = -\frac{2g}{3}\int_{\partial V} n_\alpha \epsilon^{\alpha\beta\gamma\delta}\mathrm{Tr}\left(\mathcal{A}_\beta\mathcal{A}_\gamma\mathcal{A}_\delta\right)\,.\end{aligned} \tag{1.22}$$

The reason why we cannot claim that the right hand side of (1.22) is zero is topological: there exist so-called "large" gauge transformations, that change the field as,

$$\mathcal{A}_\alpha \to \mathcal{A}_\alpha + \mathcal{D}_\alpha\Theta\,,$$

where $\mathcal{D}$ is the gauge covariant derivative, and do not vanish at infinity.

This correspond, from a topological perspective, to functions from the boundary ∂V to the gauge group, which are not continuously deformable to the identity, and if the geometry of the gauge group is not a simple circle, but for example a sphere as in the case of $SU(2)$, there can be non trivial configurations with non zero winding number. This is the reason why the right hand side of (1.22) is actually an integer, representing a winding number of how many times the group "warps" around the volume ∂V.

For these configurations the curvature vanishes, however the charge Q_5 can obtain contributions from such configurations. In the language of de Rham theory, we would be calling these closed but not exact forms, which means that we can write

$$\mathcal{A}_\alpha = \partial_\alpha\Theta\,,$$

locally, but not *globally*. These configurations are therefore topological, in the sense that they depend on the geometric relation between the gauge group and the geometry of the space in which the theory lives.

All of these remark show the importance of understanding quantum anomalies, and show how global properties of fields are intimately linked with the geometry of gauge groups and of the space in which the theory is defined. The proper understanding of these properties and relations is needed to understand the vacuum structure of the gauge theory at hand, as was shown to be the case for the baryon number and the standard model [22].

It can be argued that the same is necessary to understand the vacuum structure of gravity, whose properties are linked to the geometric structure of space-time, as Einstein's theory shows. The symmetry group of gravity, enhanced by the rescaling (1.4), has been shown to be anomalous [17, 23]. We understand part of this contribution as induced by the breaking of scale symmetry by quantum fluctuations, induced by the introduction of a renormalization scale, μ, and the consequent running of the coupling constant of the theory.

However, the Weyl anomaly also contains topological contributions in the sense of (**??**), which might be important for the definition of the vacuum structure of the theory at the critical point of the renormalization flow. In fact, near the conformal fixed point of the renormalization flow, the contributions to the Weyl anomaly that is induced by the running of the coupling constants is expected to vanish, leaving, if they exist, only the boundary contributions.

More precisely, it is not hard to see that coupling standard model matter to the metric $g_{\mu\nu}$, as to respect (1.4), yields the identity, $\langle T^{\mu}_{\mu} \rangle \Big|_g = 0$, where again the brackets $\langle \cdot \rangle \Big|_g$ indicate that the expectation value is taken in the presence of a classical field $g_{\mu\nu}$. However, explicit computations have shown that [17],

$$\langle T^{\mu}_{\mu} \rangle_g = \mathcal{A} = \Big(\alpha_{\mathcal{A}} R_{\mu\nu} R^{\mu\nu} + \beta_{\mathcal{A}} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + \gamma_{\mathcal{A}} \Box R + \zeta_{\mathcal{A}} F_{\alpha\beta} F^{\alpha\beta} \Big) \neq 0 \,, \tag{1.23}$$

where $R\,, R_{\mu\nu}\,, R_{\alpha\beta\gamma\delta}$ are the Ricci scalar, the Ricci tensor and the Riemann tensor, and $F_{\alpha\beta}$ is the field strength of a gauge field (non necessarily abelian) coupled to gravity. The coefficients $\alpha_{\mathcal{A}}\,, \beta_{\mathcal{A}}\,, \gamma_{\mathcal{A}}\,, \zeta_{\mathcal{A}}$ are universal and only depend on the number of degrees of freedom of the theory.

Because of the Gauss-Bonnet theorem, we know that part of the right hand side of (1.23) is a boundary contribution, in the sense that,

$$R^2 - 4R_{\mu\nu} R^{\nu\mu} + R_{\alpha\beta\gamma\delta} R^{\gamma\delta\alpha\beta} = \nabla_{\mu} \mathcal{V}^{\mu} \,, \tag{1.24}$$

where $\mathcal{V}^{\mu}$ is a specific combination of the space-time connection. This then induces a topological contribution to the anomaly (1.23), which can acquire boundary contributions by performing large gauge transformations, in a similar way as we demonstrated in (1.22).

An example of such large gauge transformations is given by special conformal transformations: these are a particular diffeomorphism, in flat space, with the property that,

$$x^{\mu} \to \frac{x^{\mu} - 2(x \cdot x) b^{\mu}}{1 - 2b \cdot x + b^2 x^2} \implies \eta_{\mu\nu} \to \Big(1 - 2b \cdot x + b^2 x^2 \Big)^2 \eta_{\mu\nu} \,. \tag{1.25}$$

As one can easily understand from (1.25), this transformation is singular in one point, $x^{\mu} = \frac{b^{\mu}}{b^2}$, and indeed we compute, at the end of chapter 2, the metric contribution to (1.24) and show explicitely that the integral on the boundary of space-time of the induced $\mathcal{V}$ is non zero, but equals 2.

This computation, while not being exhaustive, shows that the Weyl anomaly, similarly to the non abelian chiral anomaly, has a topological contribution. This is in some sense different from the spontaneous symmetry breaking induced by the renormalization group running of the coupling constants, and one can imagine it might play a role in the vicinity of renormalization group fixed points.

In the context of the discussion from section 1.2, the Weyl anomaly constitutes a problem: if we want to promote the Weyl symmetry to the same level as gauge transformations in quantum field theory, the associated Ward-Takahashi identities should hold. We shall see in chapter 5 that the introduction of torsion as a compensating field modifies in a non trivial way the

Ward-Takahashi identity (1.23). We will show by explicit computation that the extended Ward identities are respected by the 2 points vertices of the theory, but the torsion longitudinal polarization (which in normal gauge theories is a pure gauge contribution), develops physical excitations.

This leads to the interpretation of the Weyl anomaly as signaling the presence of additional degrees of freedom, which we identify as the space-time torsion, and are needed for the self consistency of the theory. The large gauge transformations, *i.e.* conformal transformations, we study at the end of chapter 2 can then be interpreted as large gauge transformations of the torsion field. Similar ideas have been put forward before in the literature, as we shall review shortly, however our approach is more systematic in the sense that it provides a full non linear description which seems to respect the Ward-Takahashi identities of Weyl symmetry, at least this is what we conjecture.

As was shown in [24], one cannot write a local action reproducing the right hand side of (1.23), but within some conditions, there exists a non local representation for it. One can then remove the non locality by introducing the appropriate degree of freedom, in this case a scalar field satisfying, at lowest order in the derivative expansion,

$$\Box\chi + \mathcal{O}(\partial^4)\chi = \mathcal{A}\,. \tag{1.26}$$

Furthermore, in [25], it was shown that, in QED, the 3-point function,

$$\langle T_{\mu\nu} J^\alpha J^\beta \rangle\,,\; J^\alpha = \langle \bar{\psi}\gamma^\alpha\psi \rangle\,,$$

has a massless pole, in the limit $m_\psi \to 0$, in the spectral representation, which corresponds to a physical particle, which becomes massless in the limit of massless fermions, that is when the scale symmetry is spontaneously, versus explicitely, broken.

This leads to the interpretation of the anomaly as signaling the existence of a scalar degree of freedom, gravitational in nature and massless, obeying (1.26). This represents the goldstone mode of broken scale transformations, and indeed is expected to be massless if no explicit symmetry breaking terms are included, or if the symmetry breaking happens spontaneously, via the Coleman-Weinberg condensation. If this arguments are correct, one should be, at least in principle, see the signs of this scalar degree of freedom in gravitational waves events, such as the recent black holes collisions observed by LIGO and VIRGO.

There is one important caveat in what we just described: while there is general consensus about the form of the dilaton equation of motion (1.26), it is not clear in what way the higher powers of derivatives should be included. This is because the construction of the action for the dilaton degree of freedom is not unique, but can be modified by terms which not contribute to the energy momentum tensor trace (1.23). This introduces an ambiguity in the choice of the theory to study, which will render impossible to produce reliable theoretical data needed for gravitational waves studies [2].

[2] The waveform templates used to detect gravitational signals above the noise.

We argue that a valid candidate is the theory studied in this thesis, in which the novel gravitational degrees of freedom are introduced as space-time torsion. As we demonstrate in chapter 2, introducing torsion leads to a geometric theory which is Weyl invariant, and extends the concepts of general relativity.

In chapter 5 we analyse the quantum behaviour of this theory, by computing lowest order corrections to the torsion self-energy, and the torsion graviton proper vertex. Such corrections should be proportional to the coefficient $\gamma_{\mathcal{A}}$ in (1.23), since that is the only term in the anomaly that is linear in the metric and torsion fields.

We find that the extended Ward-Takahshi identities where the torsion tensor is included are not violated at this level, even though finite contributions are gerated by the loop expansion. This happens because the extended Ward-Takahashi identities hold in any dimension, and not up to terms $\propto (D-4)$.

Furthermore, we see the appearence of a branch cut discontinuity in the longitudinal part of the torsion self energy which, in line with what we argue in chapter 3 and 4, should corresponds to a resonance of the dilaton field interacting with the scalars in the theory. The dilaton appears to be massless in the limit of massless scalar, confirming the intuition that its mass can only be generated by explicit symmetry breaking terms.

While being far from a conclusive argument [3], we can claim to have a reasonable evidence of a self consistent theory, in which the dilaton degree of freedom that the anomaly represents is correctly accounted for. Chapter 4 contains an attempt to formulate the equations one should use to compute waveforms template used in gravitational waves experiments, by analysing the theory in chapter 3 as a toy model.

1.4 Inflation, gravitational waves and the window on fundamental physics

One could speculate his or her entire life on the properties that a theory of quantum gravity might have and reach no definite conclusion. This is a question that has to be answered through experiments. The hope that particle accelerators reach the relevant length scale in the foreseable future are nihil, so one has to rely on alternative observations to probe the short distance behaviour of Nature. These can be found in cosmological and, as of recently, gravitational wave observations.

Since the universe is expanding, as time goes forward it cools down, as simple thermodynamics argument show. Therefore observing the light coming from the distant past offers a window on an universe much hotter and denser than it is today. The further away one looks, the earlier the time he or she can probe, up to the limit of photon decoupling. This corresponds to the

[3] Our computation does not capture the full right hand side of (1.23), since we limit ourselves to one functional derivative, and therefore only to the coeffient $\gamma_{\mathcal{A}}$ in (1.23).

time of recombination when the fusion of free protons and electrons into hydrogen became energetically favorable. This is the time the universe became transparent to CMB photons.

The spectrum of perturbation of this cosmic background radiation contains information about the first instants after the Big Bang, and its analysis allowed physicists to construct the standard cosmological model. According to this theory, the Big Bang was followed by a period of cosmic inflation, during which the universe expanded exponentially fast.

Reasoning on what might have been the causes of cosmic inflation has lead to a rich research, and it has become clear that what is needed to cause it, are additional degrees of freedom, either scalar fields or additional gravitational interactions. The problem is that the scarcity of data and the existence of inflationary attractors, makes many different models yield the same predictions. This problem has lead physicists to formulate inflation as an effective theory, which is based on the breaking of time translations that the expansion of the universe induces. However, no progress can be made towards fundamental understanding of inflation within this framework.

The effective theory of inflation provides a good phenomenological description, but does not explain what physical principle really caused it. A way to circumvent this is to construct theories of inflation with enhanced symmetry, which is broken today. This leads to a classification of theories based on the symmetry breaking pattern they exhibit, which will have yield similar predictions. Symmetries also impose constrains on the fields correlators, the so-called Ward identities, which can in principle be used to experimentally distinguish models with different symmetry breaking patterns. This is a non trivial task, which requires access to higher order correlators and more precise measurements. It is however, at least in principle, possible [26, 27]. The hope is that more precise measurements are coming, especially from the reionisation epoch, during which many more modes are still in their linear regime, and hence they contain much more primordial data.

In the discussion of this thesis, the symmetry to be broken is scale invariance. In chapter 3 we construct an example of such a theory, in which scale invariance gets initially spontaneously broken by the gravitational curvature, $R \neq 0$. Such a configuration has a gravitational energy, and what makes inflation possible is the transfer of this energy to the energy of the inflaton field, a scalar which is originally introduced, other than for this purpose, to make the classical action Weyl invariant.

Other than the inflaton field, we discover that the theory predicts the existence of a second scalar field, the gauge invariant combination of longitudinal mode of the torsion trace, $\mathcal{T}_\mu = \partial_\mu \chi$, and the metric scale factor. In the analysis from chapter 3, we fix the gauge to the Einstein frame, that is where the dynamical gravitons are traceless and transverse. In this gauge, the metric tensor does not contain the second scalar mode, which is therefore entirely contained in the longitudinal torsion.

Such is the Goldstone mode of broken scale transformations, which kinetically mixes with the other scalar field in a non trivial way: it modifies the internal space of the inflaton from flat to a hyperbolic one, with constant negative curvature [4].

An internal hyperbolic geometry with constant curvature is a universal feature in models with Weyl symmetry, which has been found by other authors as well [28, 29]. It is not intuitively clear why this is the case, but the consequence seems to be a suppression of tensor perturbations (*i.e.* the deviation of the metric field from the homogeneity and isotropy), and a smaller field excursion during inflation, which are both good features from the point of view of matching the observations and theoretical consistency (also in light of the so-called swampland conjecture [30]). Our discussion in chapter 3 is classical, and does not fully take into account the quantum (1-loop) effects. While we discuss these using the effective action from chapter 5, our discussion is qualitative, and we postpone a more detail study of its effect to the future.

The Goldstone mode χ can also leave an observational imprint in the cosmic microwave background, if a lot of energy is stored into its kinetic energy to begin with. This is a potential observational signature of such a theory, although it might only appear on scales much larger than the observable ones.

There is, however, another way one can try and detect a signal of the theory we develop in this thesis, which is gravitational waves observations. We stated already that breaking scale symmetry generates a Goldstone mode, which should be massless according to Goldstone theorem.

Furthermore, according to our discussion in the section 1.3, the anomaly observed in the Ward-Takahashi identities for Weyl symmetry indicates the existence of a dilaton mode, which couples to such Weyl anomaly. The terms that contribute to the anomaly are the curvature of gravitional fields, scalar condensates and possibly strong QCD condensates, where strong means of energy density comparable with the Planck energy.

These sources seem to exist in the universe: black holes mergers are accompanied by gravitational fields of the required order, and high energy QCD condensates exist in the interior of neutron stars. These sources can emit dilaton particles, in the same way they emit gravitational waves, which are in principle detectable on Earth and in the solar system. In chapter 4, we discuss how this might come about and write down the general equations one should solve to resolve the gravitational collapse where both the graviton and the dilaton are included.

This is the first step in understanding what sort of signal one can expect, that is in constructing the waveform templates used to lift the small signal detected above the noise.

Finally, by knowing how the dilaton field interacts with the Higgs particle, and other unknown scalar fields present in Nature, one could devise

[4]In scalar field theory, whose degrees of freedom are a set of scalar fields ϕ^I, the internal space metric can be read off from the kinetic term,

$$\mathcal{G}_{IJ}\partial_\mu\phi^I\partial^\mu\phi^J\,.$$

If $\mathcal{G}_{IJ} = \delta_{IJ}$ or if it can be reduced to it by a suitable field redefinition, the internal space is flat, if not we say it has curvature.

particle physics experiments, which within the next generation of observations should start putting constrains on the theory.

Chapter 2

Weyl invariant geometry in gravity with torsion

2.1 Introduction

In this chapter we want to lay down the foundations of the Weyl invariant theory that we will study throughout this thesis. This means understanding the transformations laws and the geometrical properties of tensors defined on a space-time manifold endowed with a metric compatible connection with vectorial torsion, *i.e.*

$$\Gamma^{\lambda}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\nu} + g_{\mu\nu}\mathcal{T}^{\lambda} - \delta^{\lambda}_{\nu}\mathcal{T}_{\mu}\,, \quad \Gamma^{\lambda}{}_{[\mu\nu]} = \delta^{\lambda}_{[\mu}\mathcal{T}_{\nu]}\,. \tag{2.1}$$

The introduction of vectorial torsion is necessary to "gauge" Weyl symmetry, in the sense that it will act as a gauge connection under Weyl transformations. What makes this construction interesting and non trivial is the interplay between diffeomorphisms and Weyl transformation: the gauge connection of diffeomorphisms, *i.e.* the metric, is charged under Weyl transformations, and, in a similar way, the gauge connection of Weyl transformations is charged under diffeomorphisms. This is characterised by the algebra of the enhanced diffeomorphism + Weyl symmetry group,

$$\begin{aligned}
&[\delta_{\xi}, \delta_{\theta}]g_{\mu\nu} = 2(\xi^{\lambda}\partial_{\lambda}\theta)g_{\mu\nu}\,,\; [\delta_{\xi}, \delta_{\theta}]\mathcal{T}_{\mu} = 0\,, \\
&[\delta_{\xi_1}, \delta_{\xi_2}]g/\mathcal{T} = \mathcal{L}_{[\xi_1,\xi_2]}g/\mathcal{T}
\end{aligned} \tag{2.2}$$

where $\delta_{\xi}g_{\mu\nu}/\mathcal{T}_{\mu} = \mathcal{L}_{\xi}g_{\mu\nu}/\mathcal{T}_{\mu}$ is a infinitesimal diffeomorphism acting on the metric or on the vectorial torsion ($\mathcal{L}_{\xi}$ being the Lie derivative along the vector field ξ), and, similarly, $\delta_{\theta}g_{\mu\nu} = 2\theta g_{\mu\nu}\,, \delta_{\theta}\mathcal{T}_{\mu} = \partial_{\mu}\theta$, defines our notion of infinitesimal Weyl transformation.

This Lie algebra structure, as we will see in chapter 5, will be crucial to understand the Ward identities that the theory must satisfy. For the time being, however, we will limit ourselves to a classical description, in order to understand the geometrical structure of the modified gravity theory at our hands.

In section 2.2 we consider the most general metric compatible linear connection, which depends on torsion other than the metric, and extend the transformation law (1.4) to it. We find a remarkably simple transformation that leaves unaltered the geodesic equation and the Riemann tensor. As a consequence, also the Einstein's tensor remains invariant under the complete transformation, meaning that pure gravity naturally possesses this symmetry. Since the other transformations that have the property of leaving geometry invariant are coordinate changes, and they correspond to a change in the observer's point of view, we argue that the same interpretation should be put forward for Weyl transformations such as (1.4).

We then address the question, what different physical properties do different conformal observers measure? Space-time singularities are defined through geodesic incompleteness: an observer falling towards a singularity would see its proper time stop at a certain point. Clearly, a simple rescaling of the type (1.4) can be used to extend the geodesic to arbitrary values of the proper time [9]. If one wants to follow this interpretation, he or she can ask what happens to nearby geodesics, as they get closer and closer to the singularity. We address this question by studying the geodesic deviation (Jacobi) equation, which we generalise to general space-time with torsion. We show that in a different conformal frame, observers will measure a damping force that slows the acceleration of geodesics towards each others. If the transformation really pushes the singularity to infinite proper time, this damping force becomes infinite, effectively stopping the force that pulls geodesics towards each others, when the singularity is reached. The same effect can be described in a less general setting using the Raychaudhuri equation with torsion, which is also Weyl invariant [1].

In section 2.3, we consider the interaction terms of the standard model, for fermions, gauge bosons and scalars fields. We show that our extended conformal symmetry can be easily unified with the Standard Model action, provided that the scalar field kinetic term is modified. We construct a conformally invariant covariant derivative, using the unique coordinate and metric independent contraction of torsion as the gauge field to retain local conformal invariance, that is the torsion trace. In this construction, we treat the torsion trace as a new "gauge boson" of the group defined by the transformation (1.4), motivated by the fact that the torsion trace generates local scale transformations, as can be seen in figure 1.4. We show that all interactions in the Standard Model are compatible with our version of conformal symmetry. The only way to make the gauge fields action conformally symmetric in $D \neq 4$ is to break gauge symmetry. If the gauge symmetry is Abelian and in $D = 4$ the two local symmetries can be unified at the classical level.

2.2 Conformal transformations in general relativity

In cosmology, the transformation (1.4) is often used to simplify inflationary models. For example, a complicated Lagrangian where the inflaton field is

[1]As it should be since a curve's shear, vorticity and divergence, which appear in the Raychaudhuri equation, are observable quantities, and as such should not depend on the conformal frame used to compute them.

non minimally coupled to gravity (often referred to as Jordan frame), can be studied in a more familiar setting, by performing the transformation (1.4), leading to a minimally coupled theory (in the Einstein's frame). If the transformation parameter θ in (1.4) is regular everywhere, classical solutions [2] in Jordan frame are mapped onto classical solution in Einstein frame, and the two frames are physically equivalent.

However, we can ask what different physical properties observers measure in the two frames, which are going to be the same only if the form of the equations that describe them retain their form. It becomes clear that this is not the case, if we consider the geodesic equation: since the geodesics' tangent vector squares to the rest mass of the particle, we do not expect it to retain its form when we perform a Weyl transformation. This will happen for geodesics of massless particles, obviously, but not in general. If we are after a theory which does not contain any explicit mass scale in it, geodesic' equations, as well as all other quantities, should maintain their form in every frame.

To see this explicitely, consider a local change of the metric as,

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\theta(x)} g_{\mu\nu}\,, \ \mathrm{d}\tau \to \mathrm{d}\tilde{\tau} = e^{\theta(x)}\mathrm{d}\tau\,, \tag{2.3}$$

then the Christoffel symbols, $\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}$, are shifted by,

$$\delta\overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu} = \delta^{\lambda}_{\mu}\partial_{\nu}\theta + \delta^{\lambda}_{\nu}\partial_{\mu}\theta - g_{\mu\nu}\partial^{\lambda}\theta\,. \tag{2.4}$$

It then straightforwardly follows that the geodesic equation,

$$\frac{\mathrm{d}^2 x^{\lambda}}{\mathrm{d}\tau^2} + \overset{\circ}{\Gamma}{}^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0\,, \tag{2.5}$$

transforms as,

$$e^{-\theta}\frac{\mathrm{d}}{\mathrm{d}\tilde{\tau}}\left(e^{-\theta}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tilde{\tau}}\right) + e^{-2\theta}\left(\overset{\circ}{\Gamma}{}^{\lambda}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + 2\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\tau}}\dot{x}^{\lambda} - \dot{x}^{\mu}\dot{x}_{\mu}\partial^{\lambda}\theta\right) = 0\,. \tag{2.6}$$

The third term in Eq. (2.6) can be absorbed in a reparametrization of the proper time τ, but the fourth cannot (unless $\dot{x}^{\mu}\dot{x}_{\mu} = 0$, which would represent the trajectory of a massless particle). The interpretation of this term is that it describes a force, acting on a point-like particle whose world-line is the geodesic. This force can come from scalar fields, in the case of an interaction, $\propto \phi^2 R$, or higher derivative terms, such as R^2. Clearly in the frame $\theta = const$, this force does not contribute, but in all other it will. This force is a physical property in these theories, since it comes from a physical field or higher order gravitational interactions. It's origin is the space-time variations of the Planck mass, and our claim in this chapter is that this force can be modelled

[2]Since our discussion in this chapter is fully classical, we do not look at the quantum behaviour of the theories we analyse here. We postpone the discussion on quantum effects and the well known conformal anomaly [17, 31] to Chapter 5.

as space-time torsion.

This shows that, in absence of torsion, geodesics calculated in two different conformal frames do not have, in general, the same form. Clearly this is enough to show that the geodesic equation without torsion contains, implicitely, the reference to a scale, in this case the mass of the point-like particle that is freely falling (since geodesics are, according to the Einsteinian interpretation, the trajectories of free falling bodies). Changing the way we measure this mass, in a local fashion according to (1.4), yields to the appearence of a force term in the geodesic equation. In what follows we describe how one can absorb this force into the space-time torsion, in such a way that the geodesic equation and many other geometrical equations retain their form in all frames.

It is of course possible to study a theory by doing the conformal rescaling (1.4), but one has to construct frame independent quantities that allow to relate the two frames to each others. For example, one can construct such observables in theory of cosmological perturbations [32–34], such that the observed scalar and tensor spectra do not depend on whether one calculates them in Einstein or Jordan frame. In classical General Relativity one such quantity is the Weyl tensor, the trace-free part of the Riemann tensor. Such a quantity is indeed invariant under the rescaling (1.4), however, it contains less information than the Riemann tensor itself. Namely, since the Weyl tensor is trace free, the gravitational scalars used in the Lagrangian of the theory has to be the square Weyl tensor. This leads to a theory that is substantially different from general relativity and would require a sophisticated mechanism to explain why the low energy effective theory of gravity is Einstein's theory (see for example [35] for a discussion on this class of theories). As we shall see, by adding torsion to the space-time manifold, we can construct conformally invariant theories using both the Ricci tensor or scalar. In some versions of the theory, the modified Einstein's equations are the same one studies in general relativity, safe for a field-dependent Planck mass and as such in the low energy limit they reproduce the same results as general relativity.

We now proceed to derive the transformation laws following the conformal rescaling (1.4) in Einstein-Cartan gravity, by demanding that the geodesic equation should be invariant under conformal rescaling.

Let us define the torsion tensor as the antisymmetric part of the connection,

$$T[X,Y] = -\frac{1}{2}(\nabla_X Y - \nabla_Y X - [X,Y])\,, \tag{2.7}$$

or in components

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{[\mu\nu]} = \frac{1}{2}\left(\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}\right)\,.$$

The Riemann tensor is then,

$$R[X,Y]Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,, \tag{2.8}$$

or in components,

$$R^{\lambda}{}_{\sigma\mu\nu} = \left(\partial_{\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\lambda}{}_{\sigma\mu} + \Gamma^{\lambda}{}_{\kappa\mu}\Gamma^{\kappa}{}_{\sigma\nu} - \Gamma^{\lambda}{}_{\kappa\nu}\Gamma^{\kappa}{}_{\sigma\mu}\right) . \tag{2.9}$$

We will denote vectors, forms and tensors both in their components free notation, and as their components in a local basis. Vectors, V, act on functions, f, forms, ω, act on vectors, and their action is defined as,

$$\begin{aligned} V[f] &= V^{\mu}\partial_{\mu}f \,, \text{where } f \text{ is a function} \,, \\ \omega[V] &= \omega_{\mu}V^{\mu} \,, \text{where } V \text{ is a vector} \,. \end{aligned} \tag{2.10}$$

More generally, a tensor M of rank $\binom{p}{q}$, acts linearly on p forms and q vectors to give a real number, as

$$M[\omega_1, \cdots, \omega_p, V_1, \cdots, V_q] = M^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}\omega_{\mu_1}\cdots\omega_{\mu_p}V^{\nu_1}\cdots V^{\nu_q} \,. \tag{2.11}$$

Finally, for the metric convention, we use the signature $(-,+,+,+)$.

A well known result [36] is that the most general antisymmetric connection satisfying metric compatibility is given by,

$$\Gamma^{\lambda}{}_{\mu\nu} = K^{\lambda}{}_{\mu\nu} + \mathring{\Gamma}^{\lambda}{}_{\mu\nu} = T^{\lambda}{}_{\mu\nu} + T_{\mu\nu}{}^{\lambda} + T_{\nu\mu}{}^{\lambda} + \mathring{\Gamma}^{\lambda}{}_{\mu\nu} \,, \tag{2.12}$$

where $K^{\lambda}{}_{\mu\nu}$ is often called the contorsion tensor, and $\mathring{\Gamma}^{\lambda}{}_{\mu\nu}$ are the Christoffel symbols computed using the metric, *i.e.*

$$\mathring{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{g^{\lambda\sigma}}{2}\left(\partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}\right) . \tag{2.13}$$

Let us now consider the geodesic equation: under the conformal rescaling (1.4) we have,

$$\begin{aligned} &\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tau}\nabla_{\lambda}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = 0 \to \frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tilde{\tau}}\left(\tilde{\nabla}_{\lambda}\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tilde{\tau}}\right) = \frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tilde{\tau}^2} + \tilde{\Gamma}^{\mu}{}_{\alpha\beta}\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tilde{\tau}}\frac{\mathrm{d}x^{\beta}}{\mathrm{d}\tilde{\tau}} = \\ &= e^{-2\theta}\left(\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} + (\Gamma^{\mu}{}_{\alpha\beta} + \delta\Gamma^{\mu}{}_{\alpha\beta})\dot{x}^{\alpha}\dot{x}^{\beta} - \dot{\theta}\dot{x}^{\mu}\right) = 0 \,, \end{aligned} \tag{2.14}$$

where, inspired by (2.4), we postulated that the connection transforms linearly,

$$\Gamma^{\mu}{}_{\alpha\beta} \to \tilde{\Gamma}^{\mu}{}_{\alpha\beta} = \Gamma^{\mu}{}_{\alpha\beta} + \delta\Gamma^{\mu}{}_{\alpha\beta} \,.$$

We see from Eqs. (2.4), (2.12) and (2.14) that the most natural choice is to write,

$$\delta\Gamma^{\mu}_{\alpha\beta} = \delta^{\mu}_{\alpha}\partial_{\beta}\theta \,, \qquad \delta T^{\mu}{}_{\alpha\beta} = \delta^{\mu}_{[\alpha}\partial_{\beta]}\theta \,, \qquad 2\delta T_{(\alpha\beta)}{}^{\mu} = g_{\alpha\beta}\partial^{\mu}\theta - \delta^{\mu}_{(\alpha}\partial_{\beta)}\theta \,, \tag{2.15}$$

such that the geodesic equation is mapped onto itself. The transformation law (2.15) has been considered in the literature before, for example in [18, 37], where the authors consider coupling scalar-tensor theories to torsion. In [38], the authors find the existence of an equivalent class of manifolds, analogous to a metric $e^{2\theta}\hat{g}_{\mu\nu}$, and torsion purely given by $\delta T^{\mu}{}_{\alpha\beta}$, as in (2.15), and claim that they are different representation of general relativity. We pursue such interpretation, as a manifestation of invariance of physical observables for different observers, which is reflected in the fact that the geometry remains invariant under the symmetry. For the simpler case studied in [38], the theory is indeed analogous to General Relativity, but in the general case, one has to consider the torsion trace as an external field. We show in section 2.3 that, in the classical limit, the case of pure gauge torsion (2.15) is a solution of the theory, and interpret the scalar parameter in [38] as the dilaton which sets the Planck scale.

The conformal transformations (2.15) map geodesic trajectories onto geodesic trajectories in the new frame, acting as a reparametrisation of the proper lengths. This is the case if torsion is included in the geodesic equation. In principle, one can choose not to transform the torsion tensor, and just transform the Christoffel connection as in Eq. (2.4). However, in our opinion, (2.15) is the most natural choice: namely, from differential geometry we know that a tensor W is constant along the integral curves of a vector field X if,

$$\nabla_X W = 0\,, \tag{2.16}$$

which, if the tensor acts on p 1-forms and on q vectors, transforms under conformal rescaling as,

$$(\nabla_X T + (p-q)X[\theta]T) = 0\,. \tag{2.17}$$

Requiring invariance of the parallel transport equation (2.16) leads to the following transformation laws for tensors of rank $\binom{p}{0}$ and $\binom{0}{p}$ and their covariant derivatives,

$$\begin{aligned}
&\binom{p}{0}:\ \tilde{T}^{\alpha_1\cdots\alpha_p} = e^{-p\theta}T^{\alpha_1\cdots\alpha_p}\,, \qquad \nabla_\mu T^{\alpha_1\cdots\alpha_p} \to \tilde{\nabla}_\mu \tilde{T}^{\alpha_1\cdots\alpha_p} = e^{-p\theta}\nabla_\mu T^{\alpha_1\cdots\alpha_p}\ ;\\
&\binom{0}{p}:\ \tilde{W}_{\alpha_1\cdots\alpha_p} = e^{q\theta}W_{\alpha_1\cdots\alpha_p}\,, \qquad \nabla_\mu W_{\alpha_1\cdots\alpha_p} \to \tilde{\nabla}_\mu \tilde{W}_{\alpha_1\cdots\alpha_p} = e^{q\theta}\nabla_\mu W_{\alpha_1\cdots\alpha_p}\ .
\end{aligned} \tag{2.18}$$

Following these rules, we have that all scalar contractions of tensors are invariant under conformal transformations. This choice is the most natural from the geometrical perspective, however it does not give the correct prescription when looking at fields. As an example, consider a scalar field, which under conformal transformations (1.4) changes as,

$$\phi(x) \to \tilde{\phi}(x) = e^{-\frac{D-2}{2}\theta}\phi(x)\,, \tag{2.19}$$

as can be seen from analysing the kinetic term of a scalar field theory, and the requirement that the action is dimensionless, while the geometrical prescription would give $\phi \to \phi$, since ϕ is a geometric scalar function. This does not, however, constitute a problem: scalar fields are in general different objects than geometrical scalar functions, and can therefore posses different scaling properties, even if their transformation law under coordinate transformation are the same. Scalar fields such as ϕ in (2.19) can be considered as dimensionfull quantities, such as for example the curvature scalar with torsion, R. Even though R is a scalar, it has the dimension of inverse length squared, which implies it transforms under Weyl rescalings, as $R \to e^{-2\theta} R$. Tensors can obviously have dimensions too, in which case their transformation law will differ from (2.18), but will include extra factors that come from the Weyl rescaling of their dimensionfull part.

We will see in the next section how to construct a Weyl invariant covariant derivative for a field of general conformal weight w. For the time being, we focus on tensors and forms that transform as direct product of 4-velocities, for which the property (2.18) holds. As we shall see more precisely later, these tensors, much like a direct product of 4-velocities, are dimensionless, in the sense that their entries, *e.g.* $W^{\mu_1 \cdots}_{\nu_1 \cdots}$, are pure numbers. By contrast physical fields, such as the scalar field from above, ϕ, have a physical dimension (in the scalar case it is $[E]^{\frac{D-2}{2}}$), and the most general objects living in our Weyl invariant geometry will be tensors with dimensions, examples of which are the Jacobi field, or the Riemann tensor itself.

The second main property of the connection transformation law (2.15) is that it leaves the Riemann tensor unchanged, as we can see from,

$$
\begin{aligned}
\tilde{R}^{\lambda}{}_{\sigma\mu\nu} = & \big(\partial_\mu \Gamma^\lambda_{\sigma\nu} + \delta^\lambda_\sigma \partial_\mu \partial_\nu \theta + \delta^\lambda_\sigma \partial_\mu \theta \partial_\nu \theta - \partial_\nu \Gamma^\lambda_{\sigma\mu} - \delta^\lambda_\sigma \partial_\nu \partial_\mu \theta - \delta^\lambda_\sigma \partial_\mu \theta \partial_\nu \theta \\
& + \Gamma^\lambda_{\kappa\mu} \Gamma^\kappa_{\sigma\nu} + \Gamma^\lambda_{\sigma\mu} \partial_\nu \theta + \Gamma^\lambda_{\sigma\nu} \partial_\mu \theta + \delta^\lambda_\sigma \partial_\mu \theta \partial_\nu \theta \\
& - \Gamma^\lambda_{\kappa\nu} \Gamma^\kappa_{\sigma\mu} - \Gamma^\lambda_{\sigma\nu} \partial_\mu \theta - \Gamma^\lambda_{\sigma\mu} \partial_\nu \theta - \delta^\lambda_\sigma \partial_\nu \theta \partial_\mu \theta \big) = R^{\lambda}{}_{\sigma\mu\nu} \,.
\end{aligned}
\tag{2.20}
$$

An immediate consequence of Eq. (2.20) is that the geometrical side of Einstein's equations is invariant under Weyl transformation the way we have defined them here, that is,

$$
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \to \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,.
$$

However, the right hand side of Einstein's equation, the matter side, does not have the same property. In fact, the scaling properties of the energy momentum tensor, in D dimensions, follows from its definition, if S_m is conformally invariant,

$$
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \to \tilde{T}_{\mu\nu} = e^{-(D-2)\theta} T_{\mu\nu} \,, \tag{2.21}
$$

since $\sqrt{-g}\delta g^{\mu\nu} \to e^{(D-2)\theta} \sqrt{-g} \delta g^{\mu\nu}$. The most simple way of making Einstein's equations conformally invariant, is to write a scalar field (dilaton) in Einstein's equation, playing the role of a coupling constant,

$$
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\alpha^2}{\Phi^2(x)} T_{\mu\nu} \,, \tag{2.22}
$$

where $\alpha^2 > 0$ is a dimensionless coupling constant and conformal weight of Φ is $w_\Phi = -(D-2)/2$. Equations (2.22) are conformal in any dimension, and they follow from the conformal invariant actions,

$$S_{CG} = \frac{1}{\alpha^2}\int d^D x\sqrt{-g}\Phi^2 R + S_m\,.$$

This of course implies that the Newton constant is field dependent, and its apparent value today needs to be generated by a dilaton condensate, perhaps in a similar way as the Higgs mechanism generates masses in the standard model. We will return on this issue in chapter 3, where we discuss possible mechanisms that can lead to spontaneous breaking of conformal symmetry. For the moment, let us analyse in more depth the consequences of (2.14–2.20).

2.2.1 Jacobi equation

In general relativity, the equation that describes the acceleration or deviation of nearby geodesics is the Jacobi equation,

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J = R[\dot\gamma, J]\dot\gamma\,, \tag{2.23}$$

where J are Jacobi vector fields, $\dot\gamma$ is the tangent vector to the geodesic and $R[\dot\gamma, J]\dot\gamma$ denotes the Riemann tensor.

We want to derive the equation corresponding to (2.23) in the framework of gravity with torsion and study its behaviour in different conformal frames. To this end one can select a bunch of points along the integral curves of the vector field J and look at the geodesics that start from such points. These in general relativity describe the trajectories of freely falling particles, and the geodesic deviation equation, that is the analogue of (2.23) will describe the acceleration of such test bodies towards each others due to the action of gravity.

To look at the way geodesics are pulled towards each other, we construct a variation of geodesics, that is, a set of curves $\Gamma(\tau,\sigma)$, such that for fixed σ, $\Gamma(\tau,\sigma_0)$ is a geodesic. Then we define the Jacobi field and the geodesic tangent vector as,

$$J = \left.\frac{\partial\Gamma(\tau,\sigma)}{\partial\sigma}\right|_{\tau=0},\ \dot\gamma = \left.\frac{\partial\Gamma(\tau,\sigma)}{\partial\tau}\right|_{\sigma=0}. \tag{2.24}$$

It follows from these definitions that J and $\dot\gamma$ form coordinate lines, and therefore [39],

$$\mathcal{L}_{\dot\gamma}J = [\dot\gamma, J] = 0\,,$$

where $\mathcal{L}$ is the Lie derivative, and $[\cdot,\cdot]$ denotes the commutator. We then have that the covariant derivatives of J and $\dot\gamma$ satisfy,

$$\nabla_J\dot\gamma = \nabla_{\dot\gamma}J - 2T[J,\dot\gamma]\,, \tag{2.25}$$

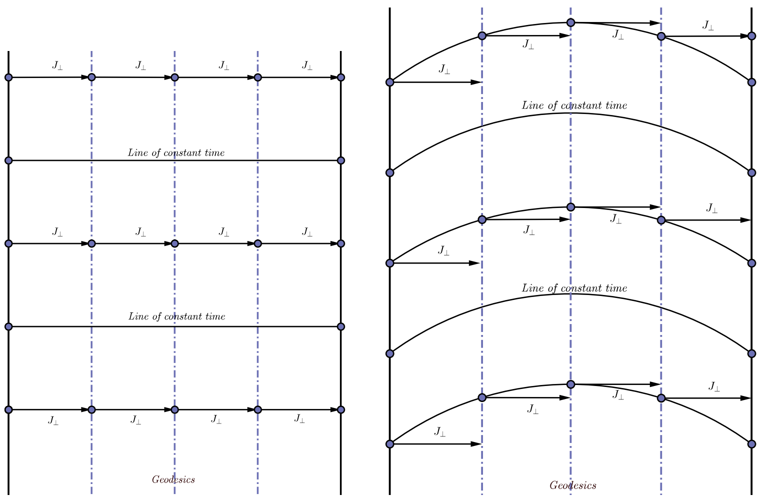


FIGURE 2.1: Geodesics in two different conformal frames for the flat plane: in both frames geodesics are straight lines, but on the right the equal time lines are bended. However, the component of $J_\perp$ remain unchanged, and are kept constant on the geodesics.

as follows straightforwardly from the torsion definition (2.7). Next, by taking the covariant derivative with respect to $\dot\gamma$ of Eq. (2.25), and applying the definition of the Riemann tensor (2.8), one finds,

$$\begin{aligned}&\nabla_{\dot\gamma}\nabla_{\dot\gamma}J - 2\nabla_{\dot\gamma}T[J,\dot\gamma] = \nabla_{\dot\gamma}\nabla_J\dot\gamma = \\ =&\nabla_J\nabla_{\dot\gamma}\dot\gamma + \left[\nabla_{\dot\gamma},\nabla_J\right]\dot\gamma = \left[\nabla_{\dot\gamma},\nabla_J\right]\dot\gamma - \nabla_{[\dot\gamma,J]}\dot\gamma = R[\dot\gamma,J]\dot\gamma\,.\end{aligned}$$

We thus find that,

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J + 2\nabla_{\dot\gamma}T[\dot\gamma,J] = R[\dot\gamma,J]\dot\gamma\,, \tag{2.26}$$

which is the Jacobi equation for space-times with torsion, and it is the correct generalisation of Eq. (2.23).

We will now demonstrate that equation (2.26) is conformally invariant. However, before we proceed, we remind that there exist two kinds of Jacobi fields: in the direction of $\dot\gamma$, there are always two linearly independent solutions, $\dot\gamma$ and $\tau\dot\gamma$, as one can easily verify by applying to them Eq. (2.26). The second fact to be noticed, is that one can project Eq. (2.26) onto the subspace orthogonal to $\dot\gamma$, since the projector operator, $h^\mu_\nu = \delta^\mu_\nu - \epsilon\dot\gamma^\mu\dot\gamma_\nu$ [3], commutes with the differential operator of Eq. (2.26). By splitting $J = (\alpha+\beta\tau)\dot\gamma + J_\perp$, one finds that $T[\dot\gamma,J] = T[\dot\gamma,J_\perp]$, and the same is true for the right-hand side,

[3]Here $\epsilon = g(\dot\gamma,\dot\gamma)$. Note that h^μ_ν is only well defined for time-like and space-like geodesics, since for null geodesics it would give the identity. This happens because h^μ_ν is degenerate for null hypersurfaces. Our construction, and the consequent Raychaudhuri equation, can be straightforwardly generalised to null geodesics, following the steps in [39].

since both $T[X,Y]$ and $R[X,Y]X$ are antisymmetric under the exchange of X and Y.

Therefore, projecting the Jacobi equation (2.26) on the subspace orthogonal to geodesics leads to,

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}J_\perp + 2\nabla_{\dot\gamma}T_\perp[\dot\gamma, J_\perp] = R_\perp[\dot\gamma, J_\perp]\dot\gamma\,, \tag{2.27}$$

where $T_\perp(\dot\gamma, J_\perp) = (h^\mu_\nu T^\nu{}_{\alpha\beta}\dot\gamma^\alpha J^\beta)\partial_\mu$, is the projection of torsion on the hyperspace perpendicular to the geodesic. Note that, since the Riemann tensor is antisymmetric in its first two indices, we have $g(\dot\gamma, R[\dot\gamma, J_\perp]\dot\gamma) = 0$, which implies that projecting the right hand side of (2.26) is irrelevant, since $R[\dot\gamma, J_\perp]\dot\gamma = R_\perp[\dot\gamma, J_\perp]\dot\gamma$.

Conformal transformations are essentially reparametrisations of the proper time, $\mathrm{d}\tau \to e^\theta \mathrm{d}\tau$, $\mathrm{d}\tau^2 = -\mathrm{d}s^2$. The integral lines of J represents lines of constant time on the neighbouring geodesics. It follows from this, that reparametrisations of proper time only change the component of J in the direction of the geodesics itself, while $J_\perp$ should stay invariant. We then postulate the following transformation laws for the Jacobi field,

$$J_\perp \;\to\; J_\perp\,, \tag{2.28}$$
$$\dot\gamma \;\to\; e^{-\theta}\dot\gamma\,, \tag{2.29}$$
$$\tau\dot\gamma \;\to\; \tilde\tau e^{-\theta}\dot\gamma\,, \tilde\tau = \int_{\tilde\tau_0}^{\tilde\tau} e^{-\theta(x(s))}\mathrm{d}s\,. \tag{2.30}$$

This is a consistent choice, because $J_\perp$ and $\dot\gamma$ are different geometrical objects: the first contains information about the separation between different freely falling observers, while the second is the four-velocity of these point-like observers. Therefore, we should not be surprised that the two vectors possess different scaling properties. However, we should notice that the magnitude of $J_\perp$ is not invariant, contrary to what happens to $\dot\gamma$. In fact, $g(J_\perp, J_\perp) \to e^{2\theta}g(J_\perp, J_\perp)$, which implies that the measured magnitude in one frame, $\|J_\perp\|$, can be arbitrarily smaller than the measured magnitude in another frame.

This result comes back to the notion of dimensionfull and dimensionless tensor fields that we mentioned above. The different transformation laws we find in Eq. (2.28) are due to the fact that $J_\perp$ has the dimension of length, as it describes the distance between neighbouring geodesics. On the other hand, $\dot\gamma$, being a 4-velocity, is dimensionless.

We can now show that the Jacobi equation (2.27) is conformally invariant. In a different frame we would write, for the left hand side of equation (2.27),

$$\tilde\nabla_{\tilde{\dot\gamma}}\tilde\nabla_{\tilde{\dot\gamma}}J_\perp + 2\tilde\nabla_{\tilde{\dot\gamma}}\tilde T_\perp[\tilde{\dot\gamma}, J_\perp] = \tag{2.31}$$
$$= e^{-\theta}\nabla_{\dot\gamma}\left[e^{-\theta}\left(\nabla_{\dot\gamma}J_\perp + \dot\theta J_\perp\right)\right] + e^{-\theta}\dot\theta\left[e^{-\theta}\left(\nabla_{\dot\gamma}J_\perp + \dot\theta J_\perp\right)\right]$$
$$+2e^{-\theta}\nabla_{\dot\gamma}\left[e^{-\theta}\left(T_\perp[\dot\gamma, J_\perp] - \frac{1}{2}\dot\theta J_\perp\right)\right] + 2e^{-\theta}\dot\theta\left(T_\perp[\tilde{\dot\gamma}, J_\perp] - \frac{1}{2}\dot\theta J_\perp\right)$$
$$= e^{-2\theta}\left(\nabla_{\dot\gamma}\nabla_{\dot\gamma}J_\perp + 2\nabla_{\dot\gamma}T_\perp[\dot\gamma, J_\perp]\right)\,, \tag{2.32}$$

and for the right hand side we have,

$$\tilde{R}[\dot{\tilde{\gamma}}, \tilde{J}_\perp]\dot{\tilde{\gamma}} = e^{-2\theta} R[\dot{\gamma}, J_\perp]\dot{\gamma}\,, \tag{2.33}$$

implying that both side of equation scale the same way, thus rendering the Jacobi equation (2.27) conformally invariant. This is precisely what we expected from the conformal invariance of the geodesic equation and of the Riemann tensor.

From the Jacobi equation (2.27), we can easily derive the Raychaudhuri equation, which has a more physically intuitive interpretation, borrowed from the context of fluid dynamics. Defining the shear, vorticity and divergence (or local expansion rate) of geodesics as,

$$S_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu \dot{\gamma}_\nu + \nabla_\nu \dot{\gamma}_\mu\right)\,, \tag{2.34}$$

$$A_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu \dot{\gamma}_\nu - \nabla_\nu \dot{\gamma}_\mu\right)\,, \tag{2.35}$$

$$\Theta = h^\mu_\nu \nabla_\mu \dot{\gamma}^\nu = \nabla_\nu \dot{\gamma}^\nu\,, \tag{2.36}$$

we can obtain the Raychaudhuri equation by isolating the $J_\perp$ dependence in (2.27). Defining the tensor $\Pi_\mu{}^\nu \equiv \nabla_\mu \dot{\gamma}^\nu$ as the covariant derivative of geodesic tangent vector, we find it satisfies,

$$\nabla_{\dot{\gamma}} \Pi_\mu{}^\nu = -\left(\Pi_\mu{}^\sigma \Pi_\sigma{}^\nu - 2T^\alpha{}_{\sigma\mu}\dot{\gamma}^\sigma \nabla_\alpha \dot{\gamma}^\nu + R^\nu{}_{\sigma\mu\lambda}\dot{\gamma}^\sigma \dot{\gamma}^\lambda\right)\,,$$

of which we can take the trace, to obtain the equation for Θ, as defined in (2.36), in terms of vorticity and shear (2.34–2.35),

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} = \left(2A_{\mu\nu}A^{\mu\nu} - 2S_{\mu\nu}S^{\mu\nu} - \frac{\Theta^2}{3} - R_{\mu\nu}\dot{\gamma}^\mu \dot{\gamma}^\nu + 2T^\alpha{}_{\beta\delta}\dot{\gamma}^\beta \nabla_\alpha \dot{\gamma}^\delta\right)\,. \tag{2.37}$$

Note that the Raychaudhuri equation (2.37) is conformal, as we can verify by using the transformation laws of vorticity, shear and local expansion rate,

$$S_{\mu\nu} \to e^\theta S_{\mu\nu}\,, \qquad A_{\mu\nu} \to e^\theta A_{\mu\nu}\,, \qquad \Theta \to e^{-\theta}\Theta\,, \tag{2.38}$$

applying the transformation law for $T^\alpha{}_{\beta\delta}$,

$$\begin{aligned} 2T^\alpha{}_{\beta\delta}\dot{\gamma}^\beta \nabla_\alpha \dot{\gamma}^\delta \to & e^{-2\theta}\left(2T^\alpha{}_{\beta\delta}\dot{\gamma}^\beta \nabla_\alpha \dot{\gamma}^\delta + \partial_\delta\theta \dot{\gamma}^\alpha \nabla_\alpha \dot{\gamma}^\delta - \dot{\gamma}^\beta \partial_\beta \theta \delta^\alpha_\delta \nabla_\alpha \dot{\gamma}^\delta\right) = \\ = & e^{-2\theta}\left(2T^\alpha{}_{\beta\delta}\dot{\gamma}^\beta \nabla_\alpha \dot{\gamma}^\delta - \frac{\mathrm{d}\theta}{\mathrm{d}\tau}\Theta\right)\,, \end{aligned} \tag{2.39}$$

and noticing that the last term in Eq. (2.39) cancels against the term coming from transforming the left hand side of Eq. (2.37), namely,

$$\frac{\mathrm{d}\tilde{\Theta}}{\mathrm{d}\tilde{\tau}} = e^{-\theta}\frac{\mathrm{d}\left(e^{-\theta}\Theta\right)}{\mathrm{d}\tau} = e^{-2\theta}\left(\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} - \frac{\mathrm{d}\theta}{\mathrm{d}\tau}\Theta\right)\,. \tag{2.40}$$

Furthermore, the fluid vorticity, shear and divergence are observables: they scale commensurably, with a conformal weight of -1. In case of global cosmological space-times (Friedmann space-times), we have that $\Theta = (D-1)H$, where H is the Hubble rate. Any cosmological measurement that intends to measure the (global) expansion rate $H(t)$ can in fact only measure the local expansion rate $\Theta(x)$ since measurement are performed locally, in the vicinity of the Earth. Θ is an observable only if geodesics and Θ are computed using the covariant derivative with torsion, which provides further theoretical support in favour of the approach proposed in this paper.

Anticipating section 2.3.3, in which we study conformal gravity endowed with a dilaton field Φ and coupled to conformal matter where we show that, when torsion is in the so-called pure gauge form, the metric $\mathrm{d}s^2$ can be written as in Eq. (2.73), such that conformal transformation of $\mathrm{d}s^2$ can be obtained by transforming the dilaton field alone as,

$$\mathrm{d}\ln(\Phi) = -\mathrm{d}\theta. \tag{2.41}$$

When this is inserted into the Raychaudhuri equation (2.37) one obtains,

$$\begin{aligned}\frac{\mathrm{d}\Theta}{\mathrm{d}\tau} &= \left(2A_{\mu\nu}A^{\mu\nu} - 2S_{\mu\nu}S^{\mu\nu} - \frac{\Theta^2}{3} - R_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} - \dot{\theta}\,\Theta\right) \\ &= \left(2A_{\mu\nu}A^{\mu\nu} - 2S_{\mu\nu}S^{\mu\nu} - \frac{\Theta^2}{3} - R_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} + \frac{\dot{\Phi}}{\Phi}\,\Theta\right)\,,\end{aligned} \tag{2.42}$$

where in the first line we took account of Eq. (2.40) and the second line is obtained from (2.41). We stress that the quantities $\Theta\,, A_{\mu\nu}\,, S_{\mu\nu}$, appearing in Eq. (2.42) are exactly mapped in the one computed in general relativity, when torsion is in the pure gauge form from Eq. (2.15). In this case Einstein's general relativity endowed with a (Brans-Dicke) scalar field corresponds to a specific gauge choice of a more general theory with torsion in which the torsion tensor can be made to vanish identically by a suitable gauge choice.

Eq. (2.42) describes the behaviour of neighbouring geodesics in different conformal frames and is used to find conditions for which singularities form [4]. First note that in this setting, if $A_{\mu\nu} = 0$ in one conformal frame, it will be zero in every frame, since it changes as $A_{\mu\nu} \to e^{\theta} A_{\mu\nu}$. Therefore vorticity cannot prevent conjugate points to form, if torsion is in its pure gauge form from Eq. (2.15). However, because of conformal invariance, we can always switch to a different frame, where the term $\propto \dot{\Phi}/\Phi$ can slow down convergence of geodesics, which might prevent the formation of conjugate points. This would mean that singularities can be moved to infinite proper time, in a different conformal frame, and if we argue that such a frame is the physical frame used by freely falling observers, we would conclude that space-time singularities cannot be reached by any physical observer.

In fact, singularities might be just bad choices of the conformal frame used: analogously to coordinate transformation, conformal transformations can be singular, and well defined only in local patches of the space-time

[4]Singularities are essentially points in which $\Theta \to -\infty$.

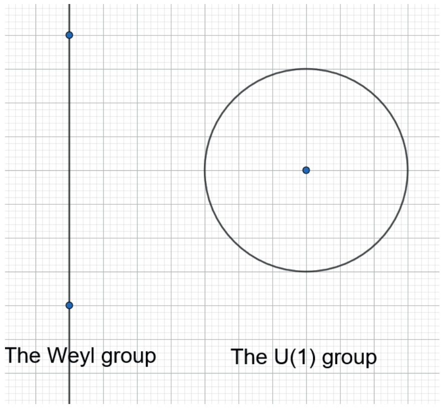


FIGURE 2.2: The global difference between the group of Weyl transformations, on the left, and the $U(1)$ group, on the right. The two groups are the same locally, but not globally, due to the compactness of the $U(1)$ transformations.

manifold (as for example the Rindler coordinates in Minkowski space). This should correspond to using the description of local observers who perceive divergent energy scale, and can therefore have access to parts of the manifold, but not to the whole space-time (as the Rindler observer cannot access the part of the manifold which is causally disconnected with him or her). However, the global geometric and conformal invariants remain locally well defined. Since all dimensionless scalars are not changed by Weyl transformations, they remain well-defined even in the case of singular conformal transformations. Note that R or any observable $\mathcal{O}$ with conformal weight $w \neq 0$ do not fit in this category, but R/Φ^2 and $\Phi^w\mathcal{O}$ (in $D = 4$) do. Clearly all dimensionful quantities can become singular after a singular conformal rescaling, but our assumption is that we cannot measure dimensionful parameters. Instead, we base our measurement on the local value of some field, which just means that we measure $\Phi^w\mathcal{O}$ rather than $\mathcal{O}$. Conformal singularities might exist and they are point in which conformally invariant ratios diverge. However, they can always be mapped onto an infinite (proper time) future or past.

2.3 Coupling to matter

In section 2.2 we have showed that gravity with torsion, in the framework of Einstein-Cartan gravity, exhibits a geometrical version of conformal invariance. Here we discuss how to construct a theory for scalar, spinor and vector fields, that exhibits the same kind of conformal invariance in arbitrary space-time dimensions.

We start by defining a 1-form, given by the trace of the torsion tensor,

$$\mathcal{T} \equiv \mathcal{T}_\mu \mathrm{d}x^\mu = \frac{2}{D-1} T^\lambda{}_{\lambda\mu} \mathrm{d}x^\mu \,. \tag{2.43}$$

Note that the torsion trace is the only one of its irreducible components that transforms under conformal transformations, and that the definition (2.43) is the unique metric independent and coordinate independent contraction of the torsion tensor one can construct. We propose treating the form (2.43) as the gauge boson of conformal transformations. This choice is motivated by the fact that a conformal transformation changes $\mathcal{T}$ as,

$$\mathcal{T} \to \mathcal{T} + \mathrm{d}\theta \,, \tag{2.44}$$

which is analogous to the way in which abelian gauge bosons transform, and by the fact that $\mathcal{T}$ acting on vectors generates scale transformations, as a consequence of parallel transport. The transformation law (2.44) has been noticed in the past, and has been tried to be used to unify gravity with electromagnetism. For example, in Ref. [37] the author considers the transformation law (2.15) and the fact that the Riemann tensor does not change upon applying it, and tries to link $\mathcal{T}$ to the gauge boson of $U(1)$. However, even if the transformation law for (2.44) is identical to the transformation law for the $U(1)$ connection, there is a key difference between the two: that the Abelian group $U(1)$ is compact. On the contrary the conformal transformations that we are studying in this paper form a non compact group, and is therefore to be distinguished from $U(1)$. The parameter θ in (1.4) is non periodic, $\theta(x) \in (-\infty, +\infty)$, while in $U(1)$ transformations one would write, for a field ψ, $\psi \to e^{iq\alpha(x)}\psi$, which shows that the space where the parameter α lives requires the identification $\alpha \sim \alpha + 2\pi$, *i.e.* it is a compact space. This is highlighed in figure 2.2.

Even though the concepts of $U(1)$ invariant derivative, and a conformally invariant derivative are distinct, the way to construct them is analogous. We have already mentioned that scalar field in D dimensions transforms as,

$$\phi \to e^{-\frac{D-2}{2}\theta}\phi \,.$$

The conformally invariant covariant derivative can therefore be expressed as,

$$\overline{\nabla}_\mu \phi = \partial_\mu \phi + \frac{D-2}{2}\mathcal{T}_\mu \phi = \partial_\mu \phi + \frac{D-2}{D-1} T^\lambda{}_{\lambda\mu}\phi \,, \tag{2.45}$$

and generalised to a field, $\mathbf{\Psi}$, of arbitrary conformal weight w as,

$$\overline{\nabla}_\mu \mathbf{\Psi} = \nabla_\mu \mathbf{\Psi} + (w_g - w)\mathcal{T}_\mu \mathbf{\Psi} \,, \tag{2.46}$$

where ∇_μ is the manifold covariant derivative, and w_g is the geometrical dimension of $\mathbf{\Psi}$, that is if $\mathbf{\Psi}$ is a $\binom{p}{q}$ tensor, $w_g = q - p$. Note that in order to be able to construct $\overline{\nabla}$ for a given field, we should know its scaling dimension, w. This is not different from the gauge derivative of fields charged under $U(1)$: in that case, one should know the hypercharge of the representation upon which the gauge derivative acts, Y, which is different for different fields. The role of hypercharge is played, for the conformal group, by the scaling dimension of fields, w.

We can think of the field $\mathbf{\Psi}$ as being a representation of the conformal extension of the Lorentz symmetry group, which is classified by its conformal weight w. Clearly $\mathbf{\Psi}$ is going to be also a representation of the Lorentz group, which will give it a "natural" conformal weight: under Lorentz transformations $\mathbf{\Psi} \to \mathbf{\Lambda}\mathbf{\Lambda}\cdots\mathbf{\Lambda}^{-1}\mathbf{\Lambda}^{-1}\cdots\mathbf{\Psi}$, where there are q $\mathbf{\Lambda}$'s and p $\mathbf{\Lambda}^{-1}$'s. Under global scale transformations we have,

$$x \to \lambda x \implies \mathbf{\Psi} \to (\mathbb{1}\lambda)(\mathbb{1}\lambda)\cdots(\mathbb{1}\lambda^{-1})(\mathbb{1}\lambda^{-1})\cdots\mathbf{\Psi} = \lambda^{(q-p)}\mathbf{\Psi}\,,$$

which sets its "Lorentz" or geometrical weight to $q-p$, when the global scaling behaviour is made local, *i.e.* $\lambda \to \lambda(x)$. We can however form composite objects out of Lorentz scalars with $w \neq 0$, and representations $\mathbf{\Psi}$ having $w = w_g$. One example of such field constructed using geometrical quantities is $\Theta^w \dot{\gamma}^\mu$, a vector with conformal weight $-w-1$. This shows that the conformal weight of fields can, in general, take any real value, which is a consequence of the non-compactness of the conformal group. Thinking back again to the $U(1)$ example: the electric charge is quantised because of the global identification $\alpha \sim \alpha + 2\pi$ [40]. This is not the case for the conformal group, whose representations can therefore possess any scaling behaviour. If w is an integer, $w - w_g$ simply represents the energy dimension, in natural units, of the field $\mathbf{\Psi}$. Fields for which $w = w_g$ are dimensionless in natural units, as for example $\dot{\gamma}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, measured in units of [space]/[time] and as such dimensionless in natural units.

By following this procedure, one can construct the covariant conformal derivative for spinor fields and vector bosons using the transformation laws,

$$\psi \to e^{-\frac{D-1}{2}\theta}\psi\,, \tag{2.47}$$

$$A_\mu \to e^{-\frac{D-4}{2}\theta}A_\mu\,. \tag{2.48}$$

The form of $\overline{\nabla}$ for gauge fields follows from Eq. (2.46), and for fermions we define,

$$\overline{\nabla}_\mu\psi = \nabla_\mu\psi + \frac{D-1}{2}\mathcal{T}_\mu\psi = \nabla_\mu\psi + T^\lambda{}_{\lambda\mu}\psi\,, \tag{2.49}$$

$$\begin{aligned}\overline{\nabla}_\mu A_\nu &= \nabla_\mu A_\nu + \frac{D-2}{2}\mathcal{T}_\mu A_\nu = \\ &= \overset{\circ}{\nabla}_\mu A_\nu + \mathcal{T}_\nu A_\mu - g_{\mu\nu}\mathcal{T}_\sigma A^\sigma + \frac{D-2}{2}\mathcal{T}_\mu A_\nu\,,\end{aligned} \tag{2.50}$$

where ∇_μ is the covariant derivative with torsion satisfying metric compatibility, $\overset{\circ}{\nabla}_\mu$ the part of the covariant derivative that depends on the metric only. Note that $\overline{\nabla}_\alpha g_{\mu\nu} = 0$, since the conformal weight of $g_{\mu\nu}$ coincides with its geometrical weight. Furthermore, it is easy to check that $\overline{\nabla}_\mu$ satisfies the

Leibniz rule, and commutes with contractions and tensor product. It is also conformally and coordinate invariant [5].

We point out that the construction that lead to Eq. (2.46) is unique if we demand non-metricity to vanish. Thus Eq. (2.46) defines a derivative operator on the manifold $\mathcal{M}$, which satisfies the basic properties of derivations and is coordinate and conformally invariant, for a field of arbitrary conformal weight w. Furthermore, Eq. (2.46) reduces to the usual covariant derivative of space-time, when it acts on a field of energy dimension 0, that is when $w = w_g$. We therefore consider it appropriate for the time being, and we will proceed, in next section, to write conformally invariant actions for scalars, fermions and gauge bosons. We will limit our discussion, for the time being, to classical theories and postpone any consideration on the quantum behaviour of the theory to chapter 5.

2.3.1 Scalars

We can clearly write the kinetic term for the scalar field with internal symmetry group G as,

$$-\frac{1}{2}\int \mathrm{d}^D x\sqrt{-g}\mathrm{Tr}\overline{\nabla}_\mu\phi\overline{\nabla}_\nu\phi g^{\mu\nu} = \\ = -\frac{1}{2}\int \mathrm{d}^D x\sqrt{-g}\mathrm{Tr}\left(\partial_\mu\phi+\frac{D-2}{2}\mathcal{T}_\mu\phi\right)\left(\partial_\nu\phi+\frac{D-2}{2}\mathcal{T}_\nu\phi\right)g^{\mu\nu}, \tag{2.51}$$

which is invariant under conformal transformations. Here $\phi = \sum_a \phi^a\lambda^a$, where λ^a are the generators of the group G of internal symmetries and the trace Tr acts in the internal group space (for a real scalar field Tr is a trivial operation). Because of this we can write the following interaction terms

$$\int \mathrm{d}^D x\sqrt{-g}\mathrm{Tr}\left(\frac{\phi^2}{2\alpha^2}R-\lambda\phi^4\right) . \tag{2.52}$$

where α and λ are coupling constants. Note that, while the first term is conformally invariant in general D, the second is only in $D = 4$. This also means that α is dimensionless in general D, while λ is dimensionless only in $D = 4$ (the canonical dimension of λ is $-(D-4)$).

In 4 space-time dimensions operators of dimension 4 in the pure gravity sector can be added to our theory without spoiling conformal invariance, and they are generic in the sense that they are always generated by quantum fluctuations [17]. For example, the following effective action,

$$\int \mathrm{d}^D x\sqrt{-g}\left(\xi_1 R^2+\xi_2 R_{\mu\nu}R^{\mu\nu}+\xi_3 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) , \tag{2.53}$$

[5] There exist other choices to construct a conformally invariant derivative, namely, using $\Gamma^\lambda{}_{\lambda\mu}/D$ or $\Gamma^\lambda{}_{\mu\lambda}$ in place of $\mathcal{T}_\mu$. However, such choices are not covariant with respect to Lorentz transformations, because the Christoffel symbols do not transform as tensors.

emerges generically when the (one-loop) quantum corrections of scalars, vectors and fermions are taken account of and it is conformally invariant in four dimensions (in the sense discussed in this paper).

Note further that the space of conformally invariant theories that one can construct using conformal symmetry in Einstein-Cartan gravity is much wider and much less constrained than conformally invariant theories containing the metric alone. There the only choice we have is to write the square of the Weyl tensor, or choose the non minimal coupling $1/\alpha^2$ between the scalar field and the Ricci curvature to be $(D-2)/[4(D-1)]$.

It is worth spending a few words to analyse what theory emerges for the Higgs particle, and how its effective low energy description can reproduce the Higgs action of the standard model. Writing $H = \sum_{a=0}^{3} H^a\sigma^a/2$, where $\sigma^a/2$ are the $SU(2)$ group generators (σ^a are here the Pauli matrices, and $\sigma^0 = \mathbb{1}$ is the group identity element), we can write the Higgs conformal-gauge derivative as,

$$D_\mu H = \partial_\mu H + \frac{D-2}{2}\mathcal{T}_\mu H - ig\sum_a W^a_\mu \sigma^a \cdot H - ig' Y B_\mu H\,, \tag{2.54}$$

where $\sigma^a \cdot H$ denotes the product in the $SU(2)$ group space, g is the weak gauge coupling constant, g' is the hypercharge gauge coupling constant, $Y = 1$ is the hypercharge of the Higgs doublet and B_μ is the (Abelian) hypercharge field. The action for the Higgs field then gets modified to,

$$\int \mathrm{d}^D x\sqrt{-g}\left[-\frac{1}{2}(D_\mu H)^\dagger D^\mu H - \lambda_H (H^\dagger H)^2 + g_{H\Phi} H^\dagger H \Phi^2 - \lambda_\Phi \Phi^4\right]\,, \tag{2.55}$$

where, in order to make the action conformally invariant in $D = 4$, we traded the Higgs mass for a dilaton field Φ. This theory can exhibit spontaneous symmetry breaking in the following sense: at high energies both $\langle H\rangle$ and $\langle\Phi\rangle$ are close to zero, as required by conformal symmetry. When the energy scale drops below a critical value, $\langle\Phi\rangle$ starts growing towards a finite value (possibly driven by the non-minimal coupling $\Phi^2 R$). This process will make the Higgs potential develop a new non trivial minimum, and as a consequence also the Higgs field will develop a vacuum expectation value.

When the coupling constants satisfy, $\lambda_H\lambda_\Phi = g^2_{H\Phi}/4$, then the effective low energy action becomes,

$$\int \mathrm{d}^D x\sqrt{-g}\left[-\frac{1}{2}(D_\mu H)^\dagger D^\mu H - \lambda_H\left(H^\dagger H - \frac{g_{H\Phi}}{2\lambda_H}\langle\Phi^2\rangle\right)^2\right]\,, \tag{2.56}$$

which leads to the Higgs *vev*, $2\langle H^\dagger H\rangle = h^2 = \frac{g_{H\Phi}}{\lambda_H}\langle\Phi^2\rangle = (246\,\mathrm{GeV})^2$. Note that this value of the Higgs *vev* produces a classical cosmological constant exactly equal to zero, *i.e.* $\Lambda \propto \left(\langle H^\dagger H\rangle - \frac{g}{2\lambda_H}\langle\Phi^2\rangle\right)^2 = 0$. Of course this does not take into account quantum effects, which can produce a non vanishing cosmological constant, also by making $\lambda_H, \lambda_\Phi, g_{H\Phi}$ run with the energy scale.

In Ref. [41] the authors consider a model similar to the one defined by (2.56), and show that it possesses inflationary solutions and exhibit late time dark energy domination. The model proposed in [41] exhibits a global scale symmetry, while the one that we are proposing in this paper makes the symmetry local by introducing a coupling between the scalar fields and torsion. If we conjecture that local conformal symmetry should be realised at high energy, the theory proposed in this paper can be seen as the UV completion of the model in [41], which could in principle explain inflation and late time dark energy, while providing an interesting framework to study the microscopic properties of gravity with torsion.

2.3.2 Fermions

It is well known that in general relativity and in general space-time dimension the kinetic term of a fermionic field can be written as,

$$\int \mathrm{d}^D x \sqrt{-g}\frac{i}{2}\left[\bar{\psi}\gamma^\mu\nabla_\mu\psi - \left(\nabla_\mu\bar{\psi}\right)\gamma^\mu\psi\right] , \tag{2.57}$$

where

$$\nabla_\mu\psi = \partial_\mu\psi - \frac{1}{8}\omega^{ab}{}_\mu[\gamma_a,\gamma_b]\psi\,, \tag{2.58}$$

$$\nabla_\mu\bar{\psi} = \partial_\mu\bar{\psi} + \frac{1}{8}\omega^{ab}{}_\mu\bar{\psi}[\gamma_a,\gamma_b]\,. \tag{2.59}$$

Here $\omega^{ab}{}_\mu$ is the spin connection defined by,

$$\omega^{ab}{}_\mu = e^a_\lambda\left(\partial_\mu e^{b\lambda} + \Gamma^\lambda{}_{\sigma\mu}e^{\sigma b}\right) \tag{2.60}$$

and $\bar{\psi}$ is defined by

$$\bar{\psi} = \psi^\dagger\tilde{\gamma}\,,$$

where $\tilde{\gamma}$ satisfies,

$$\left(\gamma^\mu\right)^\dagger = \tilde{\gamma}\gamma^\mu\tilde{\gamma}\,,$$

and it is therefore invariant under conformal transformations. This can be shown by using the definition of the γ^μ matrices, to find their scaling properties under conformal transformations,

$$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu} \implies \gamma^\mu \to e^{-\theta}\gamma^\mu\,,$$

and using the fact that the gauge parameter θ is real, we find that $\bar{\psi}$ transforms like ψ.

It would be tempting to argue that the vierbein field e^μ_a transforms in the same way as $\gamma^\mu = e^\mu_a\gamma^a$, however this would be incompatible with the assumption that the tangent space metric is parallel transported, *i.e.* $\nabla_\mu\eta_{ab} = 0$. In fact in the Cartan formalism it is the flat metric that transforms under conformal transformations, which is the only way in which the flat metric remains parallel with respect to the new connection. Indeed, if the tetrad does not transform we have,

$$\omega^a_{b\mu} \to \omega^a_{b\mu} + \delta^a_b \partial_\mu \theta\,, \tag{2.61}$$

under conformal rescaling, which immediately implies that,

$$\nabla_\mu \eta_{ab} = 0 \to \nabla_\mu \tilde{\eta}_{ab} - 2\partial_\mu\theta\,\tilde{\eta}_{ab} = 0 \implies \tilde{\eta}_{ab} = e^{2\theta}\eta_{ab}\,.$$

We speculate that the reason for this is that the manifolds that we are considering are not locally isomorphic to flat spaces, but instead to *conformally* flat spaces.

In light of this comment, we notice that we can rewrite the fermionic covariant derivative as,

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{8}\omega^{[ab]}{}_\mu\left[\gamma_a, \gamma_b\right]\psi - \frac{1}{8}\omega^{(ab)}{}_\mu\{\gamma_a, \gamma_b\}\psi\,, \tag{2.62}$$

which will lead to Eq. (2.58) in the general relativity gauge, that is where the connection $\omega^{ab}{}_\mu = 0$ is anti symmetric in $(a \leftrightarrow b)$ and the covariantly conserved metric is η_{ab}. Evaluating for the connection $\omega^{ab}{}_\mu$ as in (2.61), we get,

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{8}\omega^{[ab]}{}_\mu\left[\gamma_a, \gamma_b\right]\psi - \frac{D}{4}\partial_\mu\theta\psi\,, \tag{2.63}$$

which is conformally invariant, if the conformal weight of ψ is $w_\psi = D/4$, which we can call the geometrical weight of spinor fields (in $D = 4$, $w_\psi = 1$, since under Lorentz transformations $\psi \to \boldsymbol{\Lambda}\psi$). Notice that this derivative splits into a part proportional to the spinorial generators of Lorentz transformation, $\left[\gamma_a, \gamma_b\right]$, and a part proportional to the spinorial generators of conformal transformations, $\mathbb{1}$.

In view of (2.61), the kinetic term (2.57) is conformal in any number of dimension. However, the coupling of fermions to gauge fields can be made conformal only in four dimensions since,

$$\sqrt{-g}\bar{\psi}\gamma^\mu A_\mu \psi \to \sqrt{-g}e^{-\frac{D-4}{2}\theta}\bar{\psi}\gamma^\mu A_\mu \psi\,.$$

Also the Yukawa couplings are conformal only in four dimensions since,

$$\sqrt{-g}\phi\bar{\psi}\psi \to e^{-\frac{D-4}{2}\theta}\sqrt{-g}\phi\bar{\psi}\psi\,. \tag{2.64}$$

Hence we see that all couplings between fermions and gauge bosons or scalar fields in the Standard Model are conformal in four dimensions. In four dimensions the Standard Model Lagrangian for fermions is therefore,

$$\int \mathrm{d}^4x\sqrt{-g}\left[\frac{i}{2}\left(\bar{\psi}\gamma^\mu(\nabla_\mu + eA_\mu)\psi - (\nabla_\mu - eA_\mu)\bar{\psi}\gamma^\mu\psi\right) - g_y\phi\bar{\psi}\psi\right]\,, \tag{2.65}$$

and with $\nabla_\mu = \overset{\circ}{\nabla}_\mu$ this action is invariant both under conformal and gauge transformations. The action (2.65) can be easily generalized to non-Abelian groups G by writing $A_\mu = A^a_\mu \lambda^a$, where λ^a the suitable generators of the group and a trace is taken over the group G. A similar generalization of scalar and fermionic fields is in order, as can be found in any textbook on the Standard Model.

2.3.3 Gravity plus dilaton: a toy model

To end this section, we are going now to solve a toy version of this model in the classical limit. Namely, we assume that there is only one real scalar field, Φ, that couples non minimally to gravity and therefore sets the Planck scale. Clearly this is not the realistic situation, since there should be at least one extra scalar in the model, the Higgs field, and it is charged under $SU(2)$. However, $\Phi^2 = \sum_a (\phi^a)^\dagger \phi^a$, can be thought of as an effective sum of all scalars that non minimally couple to gravity, and at the classical level our model can therefore be realistic. For simplicity, we also assume that the only non vanishing part of torsion is its trace. Since at the level of the Ricci scalar the torsion trace and the skew symmetric part of torsion decouple, since fermions only source the skew symmetric part of torsion [42], and since the remaining irreducible part of torsion is not sourced by any matter, our considerations are general. The action then reads,

$$S[g_{\mu\nu}, \Phi, \mathcal{T}_\mu] = \int \mathrm{d}^4x \sqrt{-g} \left(\frac{\Phi^2}{2\alpha^2} R - \frac{g^{\mu\nu}}{2} \overline{\nabla}_\mu \Phi \overline{\nabla}_\nu \Phi - V(\Phi) \right) + S^{SM} , \quad (2.66)$$

where S^{SM} is the action of fermions and gauge fields which, in four dimensions, does not depend on the torsion trace.

Varying the action with respect to $\mathcal{T}_\nu$ and $g^{\mu\nu}$ leads to the classical equations of motion, which read [6],

$$\overline{\nabla}_\sigma \overline{\nabla}^\sigma \Phi = \left(\overset{\circ}{\nabla}_\sigma - \mathcal{T}_\sigma \right) \overline{\nabla}^\sigma \Phi = -\frac{1}{\alpha^2} R \Phi + \lambda \Phi^3 \quad (2.67)$$

$$(6 - \alpha^2) \Phi \overline{\nabla}_\nu \Phi = \frac{(6 - \alpha^2)}{2} \overline{\nabla}_\nu \Phi^2 = 0 \,, \quad (2.68)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{\Phi^2} \Big[\left(\overset{\circ}{\nabla}_\mu + 4\mathcal{T}_\mu \right) \overline{\nabla}_\nu \Phi^2 \quad (2.69)$$

$$- g_{\mu\nu} \left(\overset{\circ}{\nabla}_\sigma + \mathcal{T}_\sigma \right) \overline{\nabla}^\sigma \Phi^2 \Big] = \frac{\alpha^2}{\Phi^2} T^m_{\mu\nu} \,,$$

where $\overset{\circ}{\nabla}_\mu$ is the covariant derivative computed using the metric and the Christoffel symbols, and $\overline{\nabla}$ is the conformal covariant derivative from Eq. (2.45).

For our toy model, the matter stress energy tensor is going to be,

$$T^m_{\mu\nu} = \overline{\nabla}_\mu \Phi \overline{\nabla}_\nu \Phi - g_{\mu\nu} \left(\frac{1}{2} \overline{\nabla}_\sigma \Phi \overline{\nabla}^\sigma \Phi + V(\Phi) \right) + T^{SM}_{\mu\nu} \,, \quad (2.70)$$

[6] Note that the usual choice of conformally coupled scalar, $\alpha^2 = 6$ in $D = 4$ here leads to no constrain on torsion. This is so because that specific choice of α^2 leads to cancellation of all the torsion contributions in the action (2.66).

where now $T^{SM}_{\mu\nu}$ is the energy-momentum tensor fermions and gauge fields [7].

The non trivial solution of Eq. (2.68) is,

$$\mathcal{T}_\mu(x) = -\frac{1}{2}\partial_\mu \log \frac{\Phi^2(x)}{\Phi_0^2}\,, \tag{2.71}$$

where we introduced the (arbitrary) scale Φ_0^2, to make the argument of the logarithm dimensionless. It represents an arbitrary energy scale, *i.e.* the value of $\Phi(x_0)$ at some arbitrary point x_0, such that the ratio $\Phi^2(x)/\Phi_0^2$ measures the variation of the field. We have thus arrived to an equation, valid in the classical limit of the theory, which shows the connection between the intrinsic scale that an observer uses to measure its distances and the transformation law (1.4). Note that Eq. (2.71) implies that the scalar field is covariantly conformally conserved, or that,

$$\overline{\nabla}_\mu \Phi(x) = \frac{1}{2\Phi(x)}\overline{\nabla}_\mu \Phi^2(x) = 0\,. \tag{2.72}$$

If Eq. (2.71) is valid, it means that the torsion is in its pure gauge form from Eq. (2.15), which in turn implies that the metric has to be in the form $\Phi^2 \otimes \hat{g}$, where $\hat{g}$ is the metric in the General Relativity gauge (*i.e.* where the torsion trace vanishes). Because of Eq. (2.71), we can argue that the metric which is parallel transported has to be,

$$\mathrm{d}s^2 = \frac{\Phi_0^2}{\Phi^2(x)}\mathrm{d}\hat{s}^2 = \frac{\Phi_0^2}{\Phi^2(x)}\hat{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\,, \tag{2.73}$$

where Φ_0 is an integration constant with the dimension of energy and $\hat{g}_{\mu\nu}$ solves the effective equation,

$$\overset{\circ}{R}_{\mu\nu}[\hat{g}] - \frac{1}{2}\hat{g}_{\mu\nu}\overset{\circ}{R}[\hat{g}] = \frac{\alpha^2}{\Phi_0^2}\hat{T}^{SM}_{\mu\nu} - \alpha^2 \hat{g}_{\mu\nu}\Phi_0^2\left(\frac{V(\Phi)}{\Phi^4}\right)\,, \tag{2.74}$$

where $\circ$ denotes as usual quantities computed using only the metric without torsion, in this case the metric $\hat{g}_{\mu\nu}$ in Eq. (2.73), and

$$\hat{T}^{SM}_{\mu\nu} = -\frac{2}{\sqrt{-\hat{g}}}\frac{\delta S^{SM}}{\delta \hat{g}^{\mu\nu}}\,.$$

Now, we note that Φ is not a dynamical field, in fact if $\overline{\nabla}_\mu \Phi = 0$, Eq. (2.67) turns non dynamical, and it is solved by $\Phi = 0$ and, if $R > 0$, by $\Phi^2 = R/\lambda\alpha^2$. Plugging this in Eq. (2.75), leads to

$$\overset{\circ}{R}_{\mu\nu}[\hat{g}] - \frac{1}{2}\hat{g}_{\mu\nu}\overset{\circ}{R}[\hat{g}] = \frac{\alpha^2}{\Phi_0^2}\hat{T}^{SM}_{\mu\nu} - \lambda\alpha^2 \hat{g}_{\mu\nu}\Phi_0^2\,, \tag{2.75}$$

which are Einstein's equations with a positive cosmological constant. We

[7] In this toy model the Higgs contribution may (but need not) be absorbed into Φ.

notice that such a solution only exists if $R > 0$, that is in our notation de Sitter space [8], which is also supported by the results in [43] where the authors find that a condensation of the scalar field is only possible in de Sitter spacetime. However, in our theory, the restriction $R > 0$ only holds when there is only one dilaton field Φ. If more scalars are introduced, they would all turn dynamical and the space of solutions will become bigger, and not restricted to $R > 0$. Even in this situation we can solve exactly for the torsion trace, since Eq. (2.68) would still not contain kinetic terms for torsion.

The metric (2.73) clearly splits in two different representations of the symmetry group defined by (1.4): conformal transformations change Φ^{-2}, in (2.73), while Lorentz transformations only act on $\Phi_0^2 \mathrm{d}\hat{s}^2 = \Phi_0^2 \hat{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$. From this point of view, the metric is a composite object that contains two very distinct parts: a dimension-full part, Φ^{-2}, is related to the Planck mass, while the dimensionless part, $\Phi_0^2 \hat{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$, is the metric that solves Einstein's equations.

Eq. (2.73) sets the form of the metric that an observer uses to measure proper time. Now let us consider an observer performing local experiments: any measure he performs will be compared with the only local scale he observes, that is the Planck mass. However, since the Planck mass in our theory is given by the field Φ (see Eq. (2.22)), the local scale of observers is set by the field Φ itself. Note that this is precisely the interpretation of Eq. (2.73): modulo a constant proportionality factor, the natural length unit is set in this theory by the Planck scale. This is in line with our comment in the introduction: the transformation law (1.4) is really just a change of reference frame, switching from different observers that perceive locally different physical scales. Eq. (2.73) then shows that the natural scale to measure proper lengths is set by the dilaton which produces the Planck scale.

The interpretation of space-time singularities differs in this theory from the mainstream interpretation: from the form of (2.73) and (2.69) one can infer that singularities are points where $\Phi = 0$. Since $\mathrm{d}s^2$ diverges when this situation is realised (as one infers from (2.73)), it would take an infinite amount of proper time to reach such singular points. This in particular means that collapsing matter can never reach the singularity. Since physical black holes eventually evaporate, all the matter that has fallen into it will be released, once the horizon shrinks enough.

In this process, the physical separation of a congruence of geodesics, *i.e.* $\|J_\perp\| = \|\hat{J}_\perp\|/\Phi^2$ defined as in section 2.2, might not go to zero, even if $\|\hat{J}_\perp\|$ (the separation in Einstein's frame) does. When conjugate points form we have $\|J_\perp\| \to 0$, which can happen only in the asymptotic proper-time future, and might be even prevented by the dynamics of the field Φ.

2.3.4 Gauge fields

As we have remarked before, gauge fields do not transform under conformal transformations only in four dimensions. It is in fact worth noticing that

[8] Note that the de Sitter metric would be an exact solution, for the metric $\hat{g}_{\mu\nu}$, of Eq. (2.75), if $\hat{T}^{SM}_{\mu\nu} = 0$.

$D = 4$ is the only dimension in which the gauge field lagrangian can be made invariant simultaneously under conformal transformations and gauge transformations [9]. In fact, in general dimensions the conformally invariant field strength has to be written as,

$$F^a_{\mu\nu} = \overline{\nabla}_{[\mu} A^a_{\nu]} = \partial_{[\mu} A^a_{\nu]} + \frac{D-4}{2} \mathcal{T}_{[\mu} A^a_{\nu]} \,, \tag{2.76}$$

and therefore spoils gauge invariance. Conversely, if we want to write a field strength that preserves gauge symmetry, we have to neglect the second term in Eq. (2.76), which will spoil conformal symmetry.

Luckily we live in four dimensions, where the gauge field action,

$$-\frac{1}{4} \int \mathrm{d}^4 x \sqrt{-g} \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu} \right) \,, \tag{2.77}$$

is both conformally and gauge invariant. Similarly, one can show that the action of the form,

$$\int d^4 x f \mathrm{Tr}\left[F_{\mu\nu} \tilde{F}^{\mu\nu} \right] \,,$$

which is conformal (and topological). Here f is a coupling constant, $\tilde{F}^{\mu\nu} = |g|^{-1/2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the dual field strength, which under conformal transformations transforms the same way as $F^{\mu\nu}$, and $\sqrt{-g}\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Cività tensor.

2.3.5 A comparison between the breaking of Weyl symmetry and Abelian gauge symmetry

It is important to understand that there are differences between the breaking of local Weyl symmetry and that of local Abelian gauge symmetry we have already remarked in figure 2.2. The purpose of this section is to underpin the similarities and differences between the two. Our starting point is the Einstein gauge action (3.9), which we will encounter in next chapter, and is a generalization of the action (2.66), with the addition of the R^2 term and the kinetic term of $\mathcal{T}_\mu$,

$$\begin{aligned} S_E = \int \mathrm{d}^4 x \sqrt{-g} \Big[& -\left(\frac{\xi^2}{16\alpha} + \lambda \right) \left(\delta_{IJ} \phi^I \phi^J \right)^2 + \frac{\xi}{8\alpha} \omega^2 \delta_{IJ} \phi^I \phi^J \\ & + \frac{\omega^2}{2} \left(\mathring{R} + 6 \mathring{\nabla}_\mu \mathcal{T}^\mu - 6 \mathcal{T}_\mu \mathcal{T}^\mu \right) - \frac{\omega^4}{16\alpha} - \frac{1}{4} \mathcal{T}_{\mu\nu} \mathcal{T}^{\mu\nu} \\ & - \frac{1}{2} \delta_{IJ} g^{\mu\nu} \left(\partial_\mu + \mathcal{T}_\mu \right) \phi^I \left(\partial_\nu + \mathcal{T}_\mu \right) \phi^J \Big] \,, \mathcal{T}_{\mu\nu} = \partial_\mu \mathcal{T}_\nu - \partial_\nu \mathcal{T}_\mu \,, \end{aligned} \tag{2.78}$$

[9] This fact alone is of some interest, as it could be used as a starting point for the explanation of why we live in a four dimensional space-time.

and the action for the Abelian-Higgs model (also known as scalar quantum electrodynamics, or short SQED),

$$
\begin{aligned}
S_{\rm SQED} = \int {\rm d}^4x\sqrt{-g}\bigg[& -\frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\
& - g^{\mu\nu}\left\{(\partial_\mu - ieA_\mu)\varphi^*(\partial_\nu + ieA_\nu)\varphi\right\} - V(\Phi)\bigg]\,,
\end{aligned}
\tag{2.79}
$$

where e is the gauge coupling (electric charge), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength associated to the Abelian gauge field A_μ, φ is a complex scalar and

$$
V(\varphi) = -\mu^2\varphi\varphi^* + \lambda(\varphi\varphi^*)^2 \tag{2.80}
$$

is the potential, which for $\mu^2 > 0$ exhibits spontaneous symmetry breaking that 'breaks' the local gauge symmetry.

Note first that the action (2.78) is invariant (*i.e.* it transforms into itself) under local Weyl transformations,

$$
\begin{aligned}
\omega &\to \omega e^{-\zeta(x)}\,, \phi^I \to e^{-\zeta(x)}\hat{\phi}^I\,, \\
\mathcal{T}_\mu &\to \mathcal{T}_\mu + \partial_\mu\zeta(x)\,, g_{\mu\nu} \to e^{2\zeta(x)}g_{\mu\nu}\,,
\end{aligned}
\tag{2.81}
$$

where $\zeta(x)$ is an arbitrary (regular) function of space and time. This means that a suitable choice of ζ can fix ω to a nonvanishing constant, which defines the Planck scale $M_{\rm P}$. This completely fixes Weyl symmetry.

Analogously, the Abelian-Higgs action (2.79) is invariant under local gauge transformations,

$$
A_\mu \to A_\mu + \partial_\mu\tilde{\zeta}\,, \qquad \varphi \to \exp[-ie\tilde{\zeta}(x)]\varphi\,, \tag{2.82}
$$

where $\tilde{\zeta} = \tilde{\zeta}(x)$ is an arbitrary scalar gauge function.

The equation of motion for the torsion trace is obtained by varying the action (3.9),

$$
\begin{aligned}
& \mathring{\nabla}_\mu\mathring{\nabla}^\mu\mathcal{T}_\nu - \mathring{\nabla}^\mu\mathring{\nabla}_\nu\mathcal{T}_\mu - (6\omega^2 + \phi_I\phi^I)\mathcal{T}_\nu = \frac{1}{2}\partial_\nu(\phi_I\phi^I + 6\omega^2) \equiv \mathcal{J}_\nu\,, \\
& \mathcal{J}_{[\nu,\mu]} = 0\,.
\end{aligned}
\tag{2.83}
$$

Notice that the source current $\mathcal{J}_\nu$ is purely longitudinal, which is opposite to what happens in gauge theories. Indeed, the equation of motion of the gauge field implied by the action (2.79) reads,

$$
\begin{aligned}
& \mathring{\nabla}^\mu\mathring{\nabla}_\mu A_\nu - \mathring{\nabla}^\mu\mathring{\nabla}_\nu A_\mu - 2e^2\varphi\varphi^*A_\nu = ie\left[\varphi\partial_\nu\varphi^* - \varphi^*\partial_\nu\varphi\right] \equiv J_\nu\,, \\
& \mathring{\nabla}^\mu J_\mu = 0\,,
\end{aligned}
\tag{2.84}
$$

where the scalar electromagnetic current J_μ is purely transverse. (The current transversality condition is a consistency condition that can be traced back to the gauge symmetry: since a massive gauge field contains at most 3 physical degrees of freedom, the current J_μ can have at most 3 independent components.)

The first difference to notice in Eq. (2.83) and Eq. (2.84) is that, in (2.83), the effective mass does not vanish at zero scalar field condensate, $\phi_I\phi^I \to 0$. This is due to the gauge fixing condition, $\omega^2 = 4\alpha R + \xi\phi_I\phi^I \to M_P^2$, which guarantees that the curvature condensate does not vanish when $\phi_I\phi^I$ vanishes.

In order to reduce it to the backbones, let us recast the gauge field equation (2.84) in flat space (Minkowski) limit $g_{\mu\nu} = \eta_{\mu\nu}$. It is instructive to study (2.84) in the flat space-time limit, and assume that the scalar condensate is constant, such that (2.84) reduces to a Proca theory with a mass term given by,

$$M_A^2 \equiv 2e^2\langle\varphi\varphi^*\rangle \,. \tag{2.85}$$

Acting suitable derivative operators on the Proca equation (2.84) separates it into transverse and longitudinal equations as follows,

$$(\partial^2 - M_A^2)\partial_{[\mu}A_{\nu]} \;=\; \partial_{[\mu}J_{\nu]}\,, \quad J_\nu = ie\,\langle\varphi\partial_\nu\varphi^* - \varphi^*\partial_\nu\varphi\rangle \tag{2.86}$$

$$M_A^2\partial^\nu A_\nu \;=\; 0\,, \quad (\partial_\nu M_A^2 = 0)\,, \tag{2.87}$$

which tell us that (if $\partial_\nu M_A^2 = 0$) the three propagating degree of freedom are transverse, in the Lorenz sense, and massive. Notice that the Lorenz condition, $\partial^\nu A_\nu = 0$, is exact as long as $M_A^2 \neq 0$ and $\partial_\nu M_A^2 = 0$.

In the case of Weyl symmetry breaking, upon following an analogous procedure, one obtains

$$(\partial^2 - M_T^2)\partial_{[\mu}\mathcal{T}_{\nu]} = 2(\partial^2\mathcal{T}_{[\mu} - \partial_{[\mu}\partial^\lambda\mathcal{T}_\lambda)\mathcal{T}_{\nu]}\,, \tag{2.88}$$

$$-M_T^2\partial^\nu\mathcal{T}_\nu = \frac{1}{2}\partial^2\langle\Phi^2\rangle + \big(\partial^\nu\langle\Phi^2\rangle\big)T_\nu\,, \quad M_T^2 = 6M_{\rm P}^2 + \langle\Phi^2\rangle\,, \tag{2.89}$$

which tell us that the transverse modes are only sourced by higher order interactions with the longitudinal mode (*i.e.* they are not sourced at linear order), while the longitudinal degree of freedom is sourced at linear order. This is to be contrasted with the Abelian gauge theory, in which the transverse modes are sourced in the linearised theory, while the longitudinal mode is source-free at leading order.

This analysis gives a more formal justification of the *Ansatz* we used in section 3.2, $\mathcal{T}_\mu = \partial_\mu\phi^0$, and justifies our proposition to take the longitudinal component of the torsion trace as an effective scalar degree of freedom. Indeed, Eq. (2.89) tells us that the scalar ϕ^0 mixes with the other scalar degree of freedom, and does not lead to a Lorenz condition as it is the case in the massive Proca theory. This scalar is the Goldstone mode of the broken local dilatation symmetry, and mixes nonlinearly with all the scalars of the original theory according to (3.17), which is the field redefinition that diagonalizes the field space metric.

While instructive, the above analysis is not rigorous. A rigorous analysis would entail a proper (Dirac) analysis of constraints and dynamical degrees of freedom, and we leave it for future work.

Appendix: Gauss Bonnet identities and integration

In this appendix we report some results concerning the covariant derivative (2.46) that we shall use through the rest of the thesis. The first result is that the volume form's covariant derivative, much like the metric's one, is conserved [10] Since that under a Weyl transformation $\epsilon_{\mu_1\cdots\mu_D} \to e^{D\theta}\epsilon_{\mu_1\cdots\mu_D}$ and thus $w(\epsilon_{\mu_1\cdots\mu_D}) = w_g(\epsilon_{\mu_1\cdots\mu_D})$,

$$\overline{\nabla}_\alpha \epsilon_{\mu_1\cdots\mu_D} = \overset{\circ}{\nabla}_\alpha \epsilon_{\mu_1\cdots\mu_D} - DK^\lambda{}_{\mu_1]\alpha}\epsilon_{\lambda[\mu_2\cdots\mu_D} = -DK^\lambda{}_{\mu_1]\alpha}\epsilon_{\lambda[\mu_2\cdots\mu_D} = 0\,, \quad (2.90)$$

since we have, for each α,

$$\begin{aligned}\frac{1}{D!}\sum_{\sigma\in S_D}(-1)^{\mathrm{sign}\sigma}K_{\sigma\mu_{\sigma(1)}\alpha}\epsilon_{\lambda\mu_{\sigma(2)}\cdots\mu_{\sigma(D)}} =\\ = -\frac{1}{D!}\sum_{\sigma\in S_D}(-1)^{\mathrm{sign}\sigma}K_{\lambda\mu_{\sigma(1)}\alpha}\epsilon_{\sigma\mu_{\sigma(2)}\cdots\mu_{\sigma(D)}}\,.\end{aligned}$$

To convince onself of the former, note that $\lambda \neq \sigma$, owing to the antisymmetry of $K_{[\sigma\mu]\alpha}$ and the epsilon tensor. Next, the only permutations surviving are the ones where $\mu_{\sigma(1)} = \lambda$, since λ cannot appear twice as an index of the epsilon tensor. The same is true for one amongst $\mu_{\sigma(2)}\cdots\mu_{\sigma(D)}$ and σ. The last step consists in noticing that this exchange reverses the parity of the permutations, hence the "$-$" sign. We will use this fact to compute, later in this thesis, the variation of the Gauss Bonnet integral.

To this end, we also cite [44] to prove that the Gauss Bonnet integral,

$$\begin{aligned}&\int \mathrm{d}^4x\sqrt{-g}\left(\bar{R}^2 - 4\bar{R}_{\mu\nu}\bar{R}^{\nu\mu} + \bar{R}_{\alpha\beta\gamma\delta}\bar{R}^{\gamma\delta\alpha\beta}\right) = \frac{1}{8}\int \mathrm{d}^4x\;\partial_\mu\left(\sqrt{-g}\mathcal{V}^\mu\right)\\ &\mathcal{V}^\mu = 16\epsilon^{\mu\alpha\beta\gamma}\mathrm{Tr}\left\{\gamma^5\left[\mathcal{W}_\alpha\partial_\beta\mathcal{W}_\gamma - \frac{2i}{3}\mathcal{W}_\alpha\mathcal{W}_\beta\mathcal{W}_\gamma\right]\right\}, \mathcal{W}_\alpha = \frac{i}{8}\Gamma_{\mu\nu\alpha}[\gamma^\mu,\gamma^\nu]\,,\end{aligned} \quad (2.91)$$

where γ^μ, γ^5 are Dirac matrices, belonging to the 4 dimensional Clifford algebra, and the trace is taken in Spinor space. It is not difficult to see that the scaling dimension of $\mathcal{V}^\mu$ is $+4$, which makes the derivative in (2.91) conformal in 4 dimensions, or that,

$$\partial_\mu\left(\sqrt{-g}\mathcal{V}^\mu\right) = \sqrt{-g}\;\overline{\nabla}_\mu(\mathcal{V}^\mu)\,. \quad (2.92)$$

Before ending this section, we report the calculation of the contribution to the Gauss-Bonnet integral, for the pure gauge contribution given by the special conformal transformations. If we perform a large and global special

[10] This actually holds in general metric compatible torsional theory, since it only uses the antisymmetry of the contorsion tensor $K_{\alpha\beta\gamma} = -K_{\beta\alpha\gamma}$.

conformal transformation, such as,

$$\begin{aligned} x^\mu \to \frac{x^\mu - 2(x\cdot x)b^\mu}{1-2b\cdot x + b^2x^2} \implies \\ \eta_{\mu\nu} \to \left(1-2b\cdot x+b^2x^2\right)^2 \eta_{\mu\nu} \equiv \Omega^2(x)\eta_{\mu\nu}\,, \end{aligned} \tag{2.93}$$

the Christoffel symbols will be changing by,

$$\overset{\circ}{\Gamma}{}^\lambda{}_{\mu\nu} = \delta^\lambda_\mu \partial_\nu \log\Omega + \delta^\lambda_\nu \partial_\mu \log\Omega - \eta_{\mu\nu}\partial^\lambda \log\Omega\,, \tag{2.94}$$

which implies that the Gauss-Bonnet vector, $\mathcal{V}^\mu$, from Eq. (2.91) reads,

$$\mathcal{V}^\mu = -2\left(\partial^\lambda \log\Omega \partial_\lambda \log\Omega\right)\partial^\mu \log\Omega = \frac{16b^2}{\Omega^2}\left(b^\mu - b^2x^\mu\right)\,. \tag{2.95}$$

The contribution to the Gauss-Bonnet vector $\mathcal{V}^\mu$ (2.95), can be traced back to the gauge dependent term in $\mathcal{V}^\mu$,

$$\frac{2i}{3}\mathcal{W}_\alpha \mathcal{W}_\beta \mathcal{W}_\gamma\,,$$

in (2.91), since we have,

$$\partial_{[\mu}\partial_{\nu]}\log\Omega = 0\,, \partial_{[\mu}\log\Omega\partial_{\nu]}\log\Omega = 0\,,$$

and therefore the term proportional to $\partial_{[\mu}\mathcal{V}_{\nu]}$ in (2.91) vanishes.

We are now going to compute the topological charge,

$$\begin{aligned} \chi =& \frac{1}{32\pi^2}\int_V \mathrm{d}^4x\sqrt{-g}\left(R^2 - 4R_{\mu\nu}R^{\nu\mu} + R_{\alpha\beta\mu\nu}R^{\mu\nu\alpha\beta}\right) = \\ =& \frac{1}{16\pi^2}\int_{\partial V}\mathrm{d}^3\Sigma n_\mu \mathcal{V}^\mu\,. \end{aligned} \tag{2.96}$$

When b^μ is space-like, we can evaluate the integral (2.96) if we work in Euclidean coordinates, $x^E = -ix^0$, by integrating on a ball centred at the point $x^\mu = b^\mu/b^2$, which we denote as $B_R(b^\mu/b^2)$ and then take the limit of that radius to go to infinity,

$$\chi = \frac{1}{16\pi^2}\lim_{R\to\infty} 16b^2 \int_{\partial B_R(b^\mu/b^2)} \mathrm{d}^3x n_\mu \frac{b^2x^\mu - b^\mu}{\left(1-2b_\mu x^\mu + b^2x^2\right)^2} = 2\,, \tag{2.97}$$

where we used that the volume of a 3 dimensional ball of radius R, is given by $2\pi^2R^3$, and that the normal vector,

$$n^\mu_E = \frac{x^\mu}{\|x\|} \implies n^2 = 1\,.$$

We find that the contribution from special conformal transformations, given by the pure metric contribution (that is, without torsion), gives a finite contribution to the Gauss-Bonnet integral (2.91).

Notice that the right hand side of (2.97) is an even integer number, which reflects the fact that, mathematically speaking, the conformal transformations act on a specific compactification of space-time. Such compactification is, for the case of a 4 dimensional Euclidean space a 4 sphere, which has Euler characteristic 2.

Just like in the chiral $1+1$ dimensional example of chapter 5.1, where a non vanishing chiral charge was generated by a large gauge transformation, here a large Weyl transformation generates a non vanishing integer Weyl charge, which we can represent as,

$$Q_W = \int \mathrm{d}\vec{x} \, \langle D^0 \rangle \,, \tag{2.98}$$

where $D^\mu = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta T_\mu}$ is the dilatation current. For a simple scalar field theory such as (2.66), it would read,

$$Q_W^\phi = \int \mathrm{d}\vec{x} \, \langle D^0 \rangle = \left(\frac{D-2}{2} + \frac{2}{\alpha^2}(D-1) \right) \int \mathrm{d}\vec{x} \, \langle \phi \pi_\phi \rangle \,, \tag{2.99}$$

where π_ϕ is the canonical momentum associated with ϕ. Un to a constant (which might be infinite, but vanishes in any difference ΔQ_W^ϕ) due to the canonical commutation relations on constant time hypersurface, given by a volume factor, we can rewrite explicitely,

$$Q_W^\phi = \left(\frac{D-2}{2} + \frac{2}{\alpha^2}(D-1) \right) \left(\int \mathrm{d}\vec{x} \delta^3(0) + \int \mathrm{d}\vec{x} \, \langle \{ \phi \,, \pi_\phi \} \rangle \right) \,, \tag{2.100}$$

where $\{\cdot\,,\cdot\}$ denotes the anti commutator. In (2.100) the anticommutator, $\{\phi\,,\pi_\phi\}$ is a measure of the squeezing of the state: if we represent, in the phase space (ϕ, π_ϕ), the quantum state with contours of constant probability, the squeezing measures the difference from these contours from circles. Therefore, any difference $Q_W^\phi(\Sigma(t_1)) - Q_W^\phi(\Sigma(t_2))$ measures a change in squeezing of the state of the field, between the two surfaces, $\Sigma(t_1)\,,\Sigma(t_2)$ on which the Weyl charges, $Q_W^\phi(\Sigma(t_1))\,,Q_W^\phi(\Sigma(t_2))$ are defined.

We conjecture that this corresponds to the production of fundamental excitations of the dilatation current, due to the a change in vacuum configurations for the system scalar fiel plus gravity. This simple calculation presents evidence that topological terms in the Weyl anomaly, associated with Weyl symmetry breaking, have as a consequence particle's pairs creation, or at the very least a change in the mixing of the state.

To end this chapter we note that, if the connection is changed in the fashion of (2.15), the contributions to the Gauss-Bonnet integral vanish everywhere. This leads to a geometric theory that is invariant under the full conformal group, $SO(2,4)$.

Chapter 3

Inflation as spontaneous symmetry breaking of Weyl symmetry

3.1 Introduction

The phenomena known as cosmic inflation [45, 46] is one of the most studied in high energy in the modern days [47]. The leading paradigm to study it is to construct a so-called effective field theory based on the breaking of time translation symmetry induced by the expansion of the universe [48].

While this point of view provides a self consistent phenomenological description, it does not shed any light on the physical nature of the phenomena, nor it reveals any of the basic principles that rule its dynamics. A possibility to gain understanding is to realize cosmic inflation in theories with enhanced symmetry, which are broken today. This would then lead to a phase transition of sort, in which the rolling inflaton field acts as the order parameter in the symmetry breaking process.

A popular choice in this direction is to invoke scale or conformal symmetry as the one realized in the initial state of the universe, and broken during the inflationary epoch [28, 29, 41, 49–71]. This is also the approach advocated in this section, however with an important twist.

The reasons why one expects Weyl symmetry to be restored at very early times of the cosmic history are multiple. A compelling argument is given by the idea that quantum field theories at very short distances become approximately conformal and flow towards conformally invariant theories, the so-called fixed points of the renormalization group flow. Such a quantum theory can be, at least in principle, rigorously defined, which is what makes it so appealing. Another theoretical argument is provided by the simplicity that scale invariant models enjoy. Namely, the symmetry permits a handful of (local) operators that can be included, such that these theories differ from one another only by the number of degrees of freedom they possess and by their spin.

In this chapter we consider a generic choice for the scale invariant theory we defined in chapter 2, containing $\mathcal{N}$ scalar degrees of freedom ($\in O(\mathcal{N})$), and in which the global scale symmetry is promoted to a local (gauged Weyl) invariance. Our conclusion are general in the sense that they do not depend on the specific of the theory, but only upon considering a Weyl invariant theory. In order to obtain a gauge invariant action, a compensating Weyl one form is introduced in the theory, which we interpret as the torsion tensor trace (as was argued in [72]).

The longitudinal component of the compensating one form acts as the Goldstone mode of the broken symmetry, and effectively behaves as a scalar field that kinetically mixes with the other scalars in the theory. The Weyl invariance of the underlying theory enlarges the configuration space of scalar fields from the original $O(\mathcal{N})$ invariant scalars to a negatively curved, $\mathcal{N}+1$ dimensional field space, the hyperbolic space $\mathbb{H}^{\mathcal{N}+1}$. This is similar to the hyperbolic field geometry studied in [52], where the authors consider it motivated by supersymmetry. That has also been discussed in [63], where the authors prove that a maximally symmetric field space leads to universal predictions, which might be responsible for the universality properties discussed in [28], thus justifying the name α-attractors. The physical consequence of the negative curvature of the configuration space is to stretch the potential at the boundary of the field space allowing for a plateau on which a slow roll inflation is possible. In the realization of [52] the potential is stretched at large field values, $\phi \to \infty$, while in our realization the stretching occurs near the origin of the inflaton direction, *i.e.* $\phi \sim 0$. Another interesting effect a negative configuration space curvature can have was discussed in Ref. [73], where it was pointed out that a negative curvature can reduce the effective mass of fields during inflation, making them even negative, and thus qualitatively change the inflationary dynamics. This mechanism was dubbed geometric destabilization.

The additional scalar direction given by the Goldstone mode is flat in the sense that the Goldstone mode is only derivatively coupled to the $O(\mathcal{N})$ scalars, as we would expect from a Goldstone mode. If its energy density is initially big, it can dominate the preinflationary epoch with possible observable consequences. However, it does not play a dominant role in the inflationary dynamics and can thus be neglected at any later time.

Finally we briefly study quantum corrections from the matter sector, by computing the one loop quantum effective action, and conclude that the hierarchy required to obtain inflationary observables compatible with the most recent cosmological data [7] is stable. For the same reason a late time small dark energy can be expected and maintained small, thus making the tiny observed cosmological constant in this model technically natural [74].

This chapter is organized as follows: we review the Weyl invariant theory of gravity in section 3.2 and then show how the negatively curved field space geometry is induced from the requirement that the theory is Weyl invariant. In section 3.3 we study the inflationary dynamics and explain the dynamics of the flat direction χ, which constitutes the Goldstone mode of the broken symmetry. In section 3.4 we discuss the model predictions, and discuss how we could detect the Goldstone mode χ in the CMB spectrum. Finally in section 3.5 we briefly discuss the quantum corrections to the model. In the Appendix one can find a comparison between this model and the Abelian symmetry breaking phenomenon, where we highlight the differences among the two.

3.2 Weyl invariant gravity with torsion

Defining the torsion tensor as the antisymmetric part of the connection,

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{[\mu\nu]}\,, \tag{3.1}$$

one finds an exact gauge symmetry of curvature and the geodesic equation, if the metric and the torsion are transformed according to [72],

$$T^{\lambda}{}_{\mu\nu} \to T^{\lambda}{}_{\mu\nu} + \delta^{\lambda}_{[\mu}\partial_{\nu]}\theta\,, \quad g_{\mu\nu} \to e^{2\theta} g_{\mu\nu}\,. \tag{3.2}$$

That is, defining the torsion trace as,

$$\mathcal{T}_{\mu} \equiv -\frac{2}{D-1} T^{\lambda}{}_{\mu\lambda}\,, \tag{3.3}$$

one finds it transforms locally under (3.2) as a $U(1)$ gauge field,

$$\mathcal{T} \to \mathcal{T} + \mathrm{d}\theta. \tag{3.4}$$

The transformation (3.2) also realises a reparametrization of proper time, $\mathrm{d}\tau \to e^{\theta}\mathrm{d}\tau$ which leaves the geodesic equation invariant [72]. Globally, however, the group of Weyl transformations is inequivalent to that of an Abelian gauge group as the corresponding group space can be 1-to-1 mapped onto the set of real numbers and it is thus non-compact.

In Ref. [72] we also showed how to straightforwardly extend such a symmetry to the classical standard model Lagrangian, which modifies scalar kinetic terms according to the prescription [1],

$$\partial_{\mu}\phi \to (\partial_{\mu} - \Delta_{\phi}\mathcal{T}_{\mu})\phi \equiv \nabla_{\mu}\phi\,, \tag{3.5}$$

where the Δ_{ϕ} is the scaling dimension of the field ϕ. For canonically normalized scalars, it evaluates to $\Delta_{\phi} = -(D-2)/2$.

With this prescription and using the fact that the Ricci scalar with torsion transforms under (3.2) as, $R \to e^{-2\theta}R$, the Weyl invariant operators that constitute the action in four dimensions are limited to,

$$\begin{aligned} S \;=\; & \int \mathrm{d}^4x\sqrt{-g}\Big[\alpha R^2 + \zeta R_{(\mu\nu)}R^{\mu\nu} + \frac{\xi}{2}\phi^I\phi^J\delta_{IJ}R - \frac{\lambda}{4}\left(\phi^I\phi^J\delta_{IJ}\right)^2 \qquad (3.6)\\ & - \frac{1}{2}g^{\mu\nu}\delta_{IJ}\left(\partial_{\mu}\phi^I - \Delta_{\phi}\mathcal{T}_{\mu}\phi^I\right)\left(\partial_{\nu}\phi^J - \Delta_{\phi}\mathcal{T}_{\nu}\phi^J\right) - \frac{\sigma}{4}\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}\Big]\,, \end{aligned}$$

$$\mathcal{T}_{\mu\nu} \equiv \partial_{\mu}\mathcal{T}_{\nu} - \partial_{\nu}\mathcal{T}_{\mu}\,, \tag{3.7}$$

[1]Here and thereafter we use the notation ∇_{μ} to denote the covariant derivative which commutes both with diffeomorphisms and Weyl transformations (3.2). Its definition when acting on tensors can be found in [72], and it agrees with (3.5) when acts on a scalar

where $\alpha, \zeta, \xi, \lambda$ and σ are dimensionless couplings and we allow for $O(\mathcal{N})$ invariant scalars ϕ^I, with $I = 1, \cdots, \mathcal{N}$. [2]

The study of the action (3.6) is best carried out in the "Einstein gauge", that is where gravity follows the Einsteinian dynamics. Contrary to the standard procedure found in the literature, which corresponds to a field redefinition, here we simply fix the gauge symmetry of the underlying theory, thus we do not lose any dynamical information in the procedure. By definition, in the Einstein gauge, the torsion field strength coupling σ and the parameter ζ should be *zero*. Since both these operators are gauge invariant, this would mean that they are not present in any gauge.

The first choice is not necessary: since the operator $\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}$ behaves as a massive proca field, we can treat it as a matter contribution to the stress energy tensor, and choose the coupling constant σ to be of $\mathcal{O}(1)$ such that its contributions are negligible in the inflationary period. Furthermore, since the perturbation in the transverse modes of torsion, $\mathcal{T}_\mu$, decouple at linear order from tensor and scalar cosmological perturbations, we believe that neither inflationary dynamics nor the linear perturbations analysis will change whether we include, or not, the operator $\mathcal{T}_{\mu\nu}\mathcal{T}^{\mu\nu}$ in the lagrangian.

The second operator is $R_{(\mu\nu)}R^{\mu\nu}$. This operator is problematic in any model of inflation, as it contains a ghost spin-2 degree of freedom which, when treated perturbatively, can modify the power spectrum of tensors and the tensor perturbations consistency relations [75], and is generated in the 1-loop effective action. The best course of action is to treat this operator perturbatively, which is justified if the coupling constant ζ and the scale of inflation, H/M_P, are both small. This would guarantee that the corrections to our predictions from these higher order curvature operators are suppressed by $\zeta H/M_P$, which is negligible for the scale of inflation considered here $H \simeq 10^{-9} M_P$.

The remaining longitudinal component of $\mathcal{T}_\mu$ can be modeled by a real scalar field ϕ^0 as follows,

$$\mathcal{T}_\mu = \partial_\mu \phi^0 \,. \tag{3.8}$$

This choice is discussed in detail in the Appendix, in which we discuss at length the Weyl gauge fixing and explain the differences between the local Weyl symmetry and the local Abelian gauge symmetry.

[2] In principle, we could allow for a more generic field space metric, $\mathcal{G}_{IJ}$ with a different symmetry group. Simple considerations on scale invariance show, however, that such a metric can only depend on ratios ϕ^A/ϕ^B, thus lowering the symmetry to a subgroup of $O(\mathcal{N})$. Since in this chapter we follow a logic of simplicity, we are going to choose the maximally symmetric option, that is $O(\mathcal{N})$ for which $\mathcal{G}_{IJ} = \delta_{IJ}$. Quantum corrections will in general modify the kinetic term such to replace δ_{IJ} by a more general $\mathcal{G}_{IJ}$ of the form $\mathcal{G}(\phi^K\phi^L\delta_{KL}/\mu^2)\delta_{IJ}$, where μ is a renormalization scale, which still respect the $O(\mathcal{N})$ symmetry but mildly breaks the Weyl symmetry. Furthermore, one could imagine $\mathcal{G}_{IJ} = \eta_{IJ}$, having one or more time-like directions. This would however introduce ghost-like directions in field space, which would destabilize the field dynamics, and for that reason we shall not allow that possibility.

The Einstein's gauge action, which is on-shell equivalent to (3.6), reads,

$$\begin{aligned} S_E = \int d^4x\sqrt{-g}\Big[&-\Big(\frac{\xi^2}{16\alpha}+\lambda\Big)\left(\delta_{IJ}\phi^I\phi^J\right)^2 + \frac{\xi}{8\alpha}\omega^2\delta_{IJ}\phi^I\phi^J \\ &+ \frac{\omega^2}{2}\Big(\overset{\circ}{R}+6\overset{\circ}{\nabla}{}^2\phi^0-6\partial_\mu\phi^0\partial^\mu\phi^0\Big) - \frac{\omega^4}{16\alpha} \\ &- \frac{1}{2}\delta_{IJ}g^{\mu\nu}\left(\partial_\mu+\partial_\mu\phi^0\right)\phi^I\left(\partial_\nu+\partial_\nu\phi^0\right)\phi^J\Big], \end{aligned} \tag{3.9}$$

$$\begin{aligned} \equiv \int d^4x\sqrt{-g}\Big[&-\Big(\frac{\xi^2}{16\alpha}+\lambda\Big)\left(\delta_{IJ}\phi^I\phi^J\right)^2 + \frac{\xi}{8\alpha}\omega^2\delta_{IJ}\phi^I\phi^J + \frac{\omega^2}{2}\overset{\circ}{R} \\ &- \frac{\omega^4}{16\alpha} + 3\omega^2\overset{\circ}{\nabla}{}^2\phi^0 - \frac{1}{2}\mathcal{G}_{AB}g^{\mu\nu}\partial_\mu\phi^A\partial_\nu\phi^B\Big], \; A, B = 0, 1, \cdots, \mathcal{N}, \end{aligned}$$

where ω and θ are Lagrange multiplier fields, $\mathcal{G}_{AB}$ is an extended configuration space metric which includes the longitudinal torsion direction ϕ^0,

$$\mathcal{G}_{00} = 6\omega^2 + \delta_{IJ}\phi^I\phi^J, \quad \mathcal{G}_{0I} = \delta_{IJ}\phi^J = \mathcal{G}_{I0}, \quad \mathcal{G}_{IJ} = \delta_{IJ}, \quad (I, J = 1, \cdots \mathcal{N}),$$

and we have also substituted the expression for the Ricci scalar with torsion,

$$R^\lambda{}_{\alpha\sigma\beta} = \partial_\sigma\Gamma^\lambda{}_{\alpha\beta} - \partial_\beta\Gamma^\lambda{}_{\alpha\sigma} + \Gamma^\lambda{}_{\rho\sigma}\Gamma^\rho{}_{\alpha\beta} - \Gamma^\lambda{}_{\rho\beta}\Gamma^\rho{}_{\alpha\sigma}, \tag{3.10}$$

$$R = g^{\alpha\beta}R^\lambda{}_{\alpha\lambda\beta} = \overset{\circ}{R} + 6\overset{\circ}{\nabla}{}^\mu\mathcal{T}_\mu - 6g^{\mu\nu}\mathcal{T}_\mu\mathcal{T}_\nu. \tag{3.11}$$

Symbols with a $\circ$ on top in (3.9) are computed using the metric tensor only, *e.g.*

$$\overset{\circ}{\Gamma}{}^\lambda_{\mu\nu} = \frac{g^{\lambda\sigma}}{2}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right)$$

denotes the Levi-Cività connection.

Varying the action (3.9) with respect to ω yields,

$$\omega^2 \equiv 4\alpha R + \xi\phi^2, \qquad \phi^2 \equiv \delta_{IJ}\phi^I\phi^J \tag{3.12}$$

which means that everywhere in (3.9) one can exact the replacement:

$$\omega^2 \longrightarrow 4\alpha R + \xi\phi^2, \tag{3.13}$$

which will give back the original action (3.6).

The action (3.9) still contains a Weyl gauge redundancy. A convenient gauge choice is the Weyl gauge which amounts to fixing ω to some (constant) physical scale. From (3.9) we see that ω defines the Planck scale, *i.e.*

$$\omega^2 \equiv 4\alpha R + \xi\phi^2 \longrightarrow M_P^2, \tag{3.14}$$

completely fixing the Weyl gauge. Since now we have defined a reference scale, any other scale in the model can be defined with respect to that reference scale. This accords with the general notion that all physical quantities (or, equivalently, measurements) can be represented as dimensionless ratios.

Note that, because the gauge fixing condition (3.14) can be imposed only if either ϕ or R (or both) does not vanish, we have to take the initial conditions such that at least one condensate does not vanish. In other words, the conformal point at which all scalar condensates vanish is singular, and a proper discussion of its significance is beyond the scope of this work. In the construction of the mechanism analyzed in this chapter, we assume that the scalar (inflaton) field begins at its conformal point, $\langle\phi\rangle = 0$. This choice is motivated by the conformal symmetry of the UV theory. However, the initial value for the Ricci scalar is such to satisfy (3.14), which necessarily leads to $R \neq 0$. The dynamics that follow are governed by a transfer of energy (and entropy) between the space-time fluctuations and the scalar field, as it can be best understood by considering the scalar potential and noticing that the potential energy of the scalar field drops from its value at the beginning of inflation to a lower value when the field reaches the global minimum. In this sense, the symmetry breaking described in this chapter is due to the initial conditions, which are taken at the point of enhanced symmetry. Then inflation happens as a consequence of energy exchange between the gravitational field and the scalar field. While the Ricci curvature breaks conformal invariance even near $\phi = 0$, it is still conceivable that conformal symmertry is realized in the far UV, *i.e.* at scales much greater than $\omega = M_{\mathrm{P}}$. A proper understanding of the UV conformal fixed point is, however, beyond the scope of this work. [3]

Coming back to our inflationary model, the extended *configuration space metric* in (3.9) reads,

$$\mathcal{G}_{AB}\mathrm{d}\phi^A\mathrm{d}\phi^B = \left(6M_P^2 + \phi^I\phi^J\delta_{IJ}\right)(\mathrm{d}\phi^0)^2 + 2\phi_I\mathrm{d}\phi^I\mathrm{d}\phi^0 + \delta_{IJ}\mathrm{d}\phi^I\mathrm{d}\phi^J\,, \quad (3.15)$$

and has a negative configuration space Ricci curvature,

$$\mathcal{R} = -\frac{\mathcal{N}(\mathcal{N}+1)}{6M_P^2}\,,$$

which identifies it as a hyperbolic geometry $\mathbb{H}^{\mathcal{N}+1}$.

To see this more explicitly, consider the following coordinate transformations in field space. First, let us define polar coordinates for the $O(\mathcal{N})$ scalars,

$$\begin{aligned} &\phi^1 = \rho\cos\theta_1\,, \quad \phi^2 = \rho\sin\theta_1\cos\theta_2\,, \quad \phi^3 = \rho\sin\theta_1\sin\theta_2\cos\theta_3\,, \\ &\cdots\ \phi^{\mathcal{N}-1} = \rho\sin\theta_1\sin\theta_2\cdots\sin\theta_{\mathcal{N}-2}\cos\theta_{\mathcal{N}-1}\,, \\ &\qquad \phi^{\mathcal{N}} = \rho\sin\theta_1\sin\theta_2\cdots\sin\theta_{\mathcal{N}-2}\sin\theta_{\mathcal{N}-1}\,, \ \ \phi^I\phi^I = \rho^2 \implies \\ &\mathcal{G}_{AB}\mathrm{d}\phi^A\mathrm{d}\phi^B = \left(6M_P^2 + \rho^2\right)(\mathrm{d}\phi^0)^2 + 2\rho\mathrm{d}\rho\mathrm{d}\phi^0 + \left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_{\mathcal{N}-1}\right)\,, \end{aligned} \quad (3.16)$$

[3] Curiously, if initial conditions support cosmic inflation, then its geometry can be approximated by that of de Sitter space, which exhibits a global conformal symmetry SO(2,4).

where $\mathrm{d}\Omega_{\mathcal{N}-1}$ is the metric on the $(\mathcal{N}-1)$-sphere, $S^{\mathcal{N}-1}$. Finally, defining,

$$\phi^0 = \chi - \frac{1}{2}\log\left(1+\frac{\rho^2}{6M_P^2}\right), \tag{3.17}$$

brings the metric into a diagonal form,

$$\mathcal{G}_{AB}\mathrm{d}\phi^A\mathrm{d}\phi^B = \left(6M_P^2+\rho^2\right)\mathrm{d}\chi^2 + \frac{6M_P^2\mathrm{d}\rho^2}{6M_P^2+\rho^2} + \rho^2\mathrm{d}\Omega_{\mathcal{N}-1}, \tag{3.18}$$

which can be further simplified by redefining,

$$\begin{aligned} &\frac{\mathrm{d}\rho}{\sqrt{6M_P^2+\rho^2}} = \mathrm{d}\psi \implies \rho = \sqrt{6}M_P\sinh(\psi), \\ &\implies \mathcal{G}_{AB}\mathrm{d}\phi^A\mathrm{d}\phi^B = 6M_P^2\left[\mathrm{d}\psi^2 + \left(\cosh^2(\psi)\mathrm{d}\chi^2 + \sinh^2(\psi)\mathrm{d}\Omega_{\mathcal{N}-1}\right)\right]. \end{aligned} \tag{3.19}$$

This form of the metric makes it explicit what components of the fields ϕ^I are the Goldstone modes and which are the directions acquiring a *vev*, the latter being identified with the direction ψ which, as we show in the next section, is the inflaton direction.

3.3 Inflationary dynamics

In this section we study inflation in the model presented in the previous section and show that one can obtain an inflationary model that conforms with all existing data.

In the diagonal field coordinates (3.18) the gauge fixed action (3.9) specialized to $\mathcal{N}=1$ case [4], reads,

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\Bigg[&-\left(\frac{9\xi^2}{4\alpha}+36\lambda\right)M_P^4\sinh^4(\psi) + \frac{3\xi}{4\alpha}M_P^4\sinh^2(\psi) \\ &-\frac{M_P^4}{16\alpha} + \frac{M_P^2}{2}\mathring{R} - \frac{6M_P^2}{2}g^{\mu\nu}\left[(\partial_\mu\psi)(\partial_\nu\psi) + \cosh^2(\psi)(\partial_\mu\chi)(\partial_\nu\chi)\right]\Bigg]. \end{aligned} \tag{3.20}$$

Let us begin the analysis by recalling the background cosmological metric,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}, \tag{3.21}$$

[4] In this work we focus to study only the simplest O(1) case of a real scalar field. The more general $O(\mathcal{N})$ case contains $\mathcal{N}-1$ Goldstones, and we postpone the study of their effect onto the inflationary dynamics to future publication. It is well known that the dynamics of multi-field Goldstone-like inflationary fields can produce interesting effects, see *e.g.* Ref. [76]. For now it suffices to note that, when the angular velocities are small, $|\dot{\theta}_I| \ll M_P$ ($I = 1, \cdots \mathcal{N}-1$), we expect the effect of Goldstones to be unimportant during inflation.

where $a = a(t)$ denotes the scale factor. For simplicity the spatial sections in (3.21) are assumed to be flat, *i.e.* $\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} = \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$ $(i,j = 1,2,3)$.

The equations of motion for the background (homogeneous) fields, $\psi = \psi(t)$ and $\chi = \chi(t)$, are best analyzed in terms of e-folding time, $\mathrm{d}N = H\mathrm{d}t$, where $H = \dot{a}/a$ is the Hubble parameter and $\dot{a} \equiv da/dt$. In e-folding time the scalar equation of motion and the Friedman equations decouple, that is defining the principal (first) slow roll parameter as $\epsilon = -\frac{\mathrm{d}}{\mathrm{d}N}\log H \equiv -(\log H)'$, we find,

$$\epsilon_1 \equiv \epsilon \;=\; -\frac{H'}{H} = 3\left[(\psi')^2 + \cosh^2(\psi)(\chi')^2\right]\,,\ H^2 = \frac{V(\psi)}{(3-\epsilon)M_{\mathrm{P}}^2}\,, \tag{3.22}$$

$$\frac{\psi''}{3-\epsilon} \;+\; \psi' + \frac{M_{\mathrm{P}}^2}{6}\frac{\partial\log V(\psi)}{\partial\psi} = 0\,,\quad \left[He^{3N}\cosh^2(\psi)\chi'\right]' = 0\,, \tag{3.23}$$

a *prime* ($'$) denotes a derivative with respect to N and

$$V(\psi) = M_{\mathrm{P}}^4\left[\left(\frac{9\xi^2}{4\alpha} + 36\lambda\right)\sinh^4(\psi) - \frac{3\xi}{4\alpha}\sinh^2(\psi) + \frac{1}{16\alpha}\right]. \tag{3.24}$$

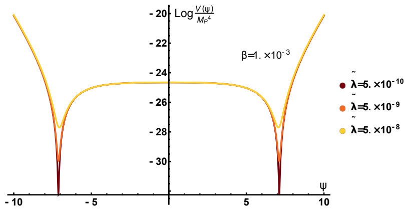
The potential (3.24) possesses a nearly flat region near the point of enhanced symmetry, $\psi = 0$ (see figure 3.1a), which is instrumental for prolongation of the inflationary phase. An analogous potential that is not however motivated by conformal symmetry is plotted on figure 3.1b for the same choice of the couplings. Note that the minimum of the potential in figure 3.1a is at a lower value of the field, implying that the models with a negative configuration space curvature roll typically over smaller distances. This feature is due to the sudden curving of the potential that can be seen in figure 3.1a induced by the negative configuration space curvature. The main consequence of such a sharp turn of the potential is, as we shall demonstrate later, a lower value of the slow roll parameters in the window $N = 50-60$ e-foldings before the end of inflation. This translates into a lower value of the tensor to scalar ratio, $r \simeq 16\epsilon_1$, which renders our model viable for a wider range of the parameter ξ as compared to the flat geometry case. In addition, due to the smaller field excursion in the hyperbolic geometry case ($\mathcal{O}(1)m_{\mathrm{P}} = \mathcal{O}(1)\sqrt{8\pi}M_{\mathrm{P}}$), it may be therefore easier to tame the Planck scale operators and the infamous η problem may be less severe, or even absent, in models with a negative configuration space curvature.

The potential (3.24) has a maximum at $\psi = 0$, at which the potential energy equals,

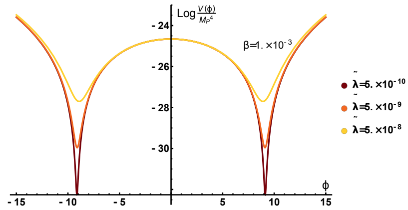
$$V(0) = \frac{M_{\mathrm{P}}^4}{16\alpha}\,, \tag{3.25}$$

and two symmetric minima,

$$\pm\,\psi_m = \pm\ln\left[\iota + \sqrt{\iota^2+1}\right]\,,\qquad \text{with}\quad \iota = \sqrt{\frac{\xi}{6(\xi^2+16\alpha\lambda)}}\,, \tag{3.26}$$



(A) Logarithm of the potential in (3.20), for various values of λ.



(B) Logarithm of the same potential for a flat field configuration space.

FIGURE 3.1: The figure illustrates how the curved geometry of field space allows for a large flat plateau near the origin, where inflation may take place. The flat region is a consequence of the negatively curved geometry of field space.

at which the potential minimizes at a value,

$$V_m = V(\pm\psi_m) = \frac{\lambda}{\xi^2 + 16\alpha\lambda} M_{\rm P}^4 \,, \tag{3.27}$$

such that, as the field rolls down from its maximum at $\psi \approx 0$ to its minimum, the potential energy density changes by, $\Delta V = -M_{\rm P}^4/[(\lambda/\xi^2) + (16\alpha)^{-1}]$. From (3.27) we see that the potential energy at the end of inflation is positive and potentially large. Indeed, unless the coupling constant λ is extremely small (and/or ξ, α extremely large), the energy density (3.27) will be much larger than the corresponding density in dark energy. Therefore, the amount in (3.27) ought to be compensated by an almost equal, negative contribution. In fact, such a negative contribution exists in the standard model. Indeed, it was pointed out in Ref. [74] that such a fine tuning can be exacted by adding both the negative contributions generated at the electroweak scale by the Higgs field condensate and the chiral condensate generated at the quantum chromodynamic (QCD) transition. In the same reference it was observed that, once this fine tuning is done at one renormalization scale, it will remain stable under an arbitrary change of the renormalization scale, which is due to the technical naturalness that arises from the enhanced symmetry at the point of vanishing vacuum energy.

A nontrivial consequence of Weyl symmetry, is the absence of any potential for the χ field. Since this is a remnant of a broken local symmetry, when quantum effects are included the χ-flatness must be preserved to all orders in perturbation theory. Namely, quantum effects can (and will) modify the term multiplying $(\partial\chi)^2$ and they can generate non-local terms, but they can never generate a potential (zero derivative) term.

An important consequence of the χ-flatness is the conservation of its canonical momentum, $\pi_\chi = He^{3N}\cosh^2(\psi)\chi'$ implied by the equation of motion for χ in (3.22). [5] This then means that,

$$\chi' = \frac{c}{He^{3N}\cosh^2(\psi)}\,, \tag{3.28}$$

where c is a constant with the meaning of field space angular momentum. If c is very large, it can change the initial dynamics, producing a period of *kination* – defined as the cosmological epoch during which kinetic energy of a scalar field dominates the Universe's dynamics [77, 78] – during which the matter fluid is characterized by the equation of state, $P = \rho$. Since χ' scales as $1/a^3$ and moreover it is proportional to $1/\cosh^2\psi$, which also decays, any c contribution to the Universe's energy density will rapidly dilute and the Universe will quickly enter a slow roll inflationary regime governed by the evolution of ψ. Nevertheless, kination should not be readily dismissed. Indeed, as we argue at the end of section 3.4, a pre-inflationary period of kination may produce interesting observable effects in the cosmic microwave background on very large angular scales, at least if inflation does not last for too long.

3.4 Results

In this section we show the results that a numerical analysis of Eqs. (3.22–3.23) yields. There are 4 free parameters that control the dynamics of this theory - 3 coupling constants α, ζ, λ and the field-space angular momentum c. In this section we will see what role each of them plays in the predictions of the model.

We start with α, as its role is simplest to understand. If we extract α out of all terms in the potential (3.24) we get,

$$V(\psi) = M_{\rm P}^4\left[\left(\frac{9\xi^2}{4\alpha} + 36\lambda\right)\sinh^4(\psi) - \frac{3\xi}{4\alpha}\sinh^2(\psi) + \frac{1}{16\alpha}\right]\,, \tag{3.29}$$

and looking at (3.23), we see that – for fixed ξ and $\tilde\lambda \equiv \alpha\lambda$ – the coupling α controls the size of the potential, *cf.* also (3.25), and thus the Hubble parameter at the beginning of inflation, as well as the scalar and tensor spectra of cosmological perturbations (recall that the corresponding amplitudes are, $\Delta_s^2 \simeq H^2/(8\pi^2\epsilon M_{\rm P}^2)$ and $\Delta_t^2 \simeq 2H^2/(\pi^2 M_{\rm P}^2) \simeq 16\epsilon\Delta_s^2$, respectively) and thus can be fixed by requiring the COBE normalization of the scalar power spectrum, $\ln(10^{10}\Delta_{s*}^2) = 3.089 \pm 0.036$.

The role of ξ can be understood from the requirement that the potential should have a sufficiently large flat region around the origin. From Eqs. (3.26) and (3.27) we see that the minima will be at a large (super-Planckian) value if $\iota \gg 1$. In this case, inflation will be long (the total number of e-foldings $N_{\rm tot}$ will be much larger than the required 60) and these models are *large field inflationary models*. In the opposite limit when $\iota \ll 1$, $\psi_m \ll 1$, inflation will

[5] An analogous conservation of the angular momentum of the Goldstone modes governs their dynamics, as each of the Goldstones exhibits a flat direction as well.

typically be short ($N_{tot} \ll 60$) and one gets a *small field inflation*. Obviously, viable inflationary models must satisfy $N_{tot} \geq 60$ and belong to large field models, for which $\iota \gg 1$. As we will see below, it is also typically the case that $16\alpha\lambda \ll \xi^2$, such that $\iota \simeq 1/\sqrt{6\xi}$, and (3.26) yields $\psi_m \simeq \ln(2\iota) \simeq \frac{1}{2}\ln(2/(3\xi))$. In figure 3.2 we illustrate the two relevant cases, large field (left panel) and small field (right panel) inflationary potentials. Upon solving the

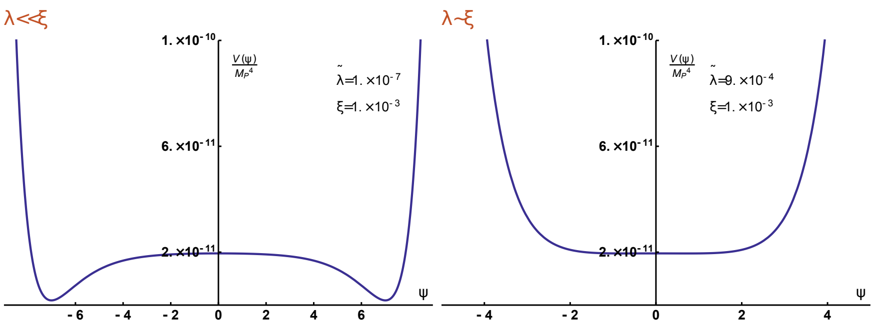


FIGURE 3.2: The hierarchy effect between $\tilde{\lambda} = \alpha\lambda$ and ξ^2. *Left panel.* There are deep minima at $\psi = \pm\psi_m$, $\psi_m \gg 1$ when the hierarchy $1 \gg \xi^2 \gg \tilde{\lambda}$ is observed. *Right panel.* When the hierarchy $\xi^2 \gg \tilde{\lambda}$ is not observed, the minima get close to the origin and become very shallow, almost unobservable by eye, although they are still present.

background equations of motion, one can confirm that increasing ξ (for fixed λ and α), decreases the duration of inflation, as can be seen in figure 3.3.

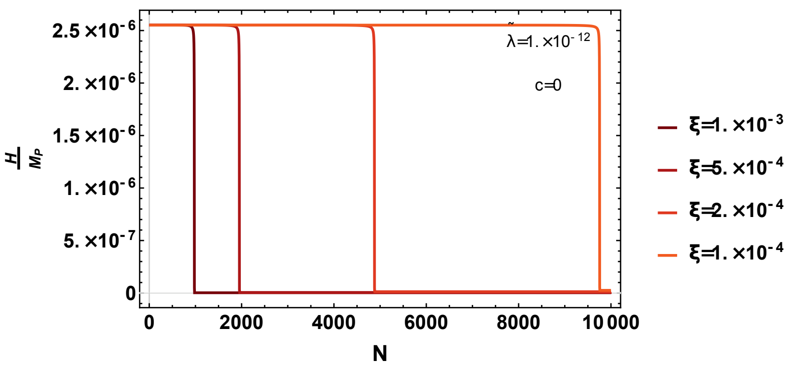


FIGURE 3.3: Increasing ξ (for fixed α and λ) decreases the duration of inflation.

To summarize, the following picture has emerged:

1) The parameter $\lambda \ll 1$ controls the vacuum energy density at the end of inflation. More precisely, from Eq. (3.27) it follows that $\lambda/(\xi^2 +$

$16\alpha\lambda) = V_m/M_P^4$ controls the post-inflationary vacuum energy density, which in the limit when $\xi^2 \gg 16\alpha\lambda$ reduces to λ/ξ^2. This vacuum energy should be of the order of the electroweak energy density, $\rho_{\rm EW} \sim 10^{-66}\, M_P^4$, implying that λ ought to be extremely small, *i.e.* $\lambda \sim 10^{-66}\xi^2$.

2) The parameter $\alpha \gg 1$ controls the amplitude of the scalar and tensor power spectra, as it fixes the value of H at the beginning of inflation. It is therefore fixed by the COBE normalization of the amplitude of scalar cosmological perturbations to be about, $\alpha \sim 10^9$, where to get the estimate we took, $r = 16\epsilon \simeq 0.003$ of the Starobinsky model.

3) The parameter $\xi \ll 1$ controls the duration of inflation, such that a small ξ implies a long inflation (large field model); a large ξ implies short inflation (small field model), see figure 3.4. More precisely, it is actually $\iota^{-2} = 6(\xi^2 + 16\alpha\lambda)/\xi$ that controls the duration of inflation, which in the limit when $\xi^2 \gg 16\alpha\lambda$ reduces to 6ξ.

Finally, one can argue that $16\alpha\lambda \ll \xi^2$ as follows. If this condition were not met, would imply (from (3.27)) a vacuum energy density at the end of inflation, $V_m \simeq (M_P^4/16\alpha)\left[1 - \xi^2/(16\alpha\lambda)\right]$, which is comparable to the vacuum energy at the beginning of inflation given by (3.25). If this energy is to be almost compensated by the vacuum energy from the electroweak symmetry breaking, this would mean that inflation would have to happen at the electroweak scale. Inflation at the electroweak scale is possible, but it is much more fine tuned than inflation close to the grand unified scale, at which $H \sim 10^{13}$ GeV, and thus theoretically disfavored.

3.4.1 Model predictions for n_s and r

In what follows, we present inflationary predictions of our model in slow roll approximation [6]. There are essentially four observables [7] from the Gaussian cosmological perturbations - from which two have been observed and for the other two there are limits and there are only limits on non-Gaussianities. The scalar and tensor spectrum can be written as,

$$\Delta_s^2(k) = \Delta_{s*}^2\,(k/k_*)^{n_s-1}\,, \qquad \Delta_t^2(k) = \Delta_{t*}^2\,(k/k_*)^{n_t} \tag{3.30}$$

where Δ_{s*}^2 is the amplitude of the scalar spectrum, which is fixed by the COBE normalization,

$$A_s \equiv \Delta_{s*}^2 = (2.105 \pm 0.030) \times 10^{-9} \tag{3.31}$$

and n_s is the spectral slope defined such that, when $n_s = 1$, the scalar spectrum is scale invariant. Planck (and other available) data [7] constrain n_s at

[6] By this we mean that the formulas for n_s, α_s and r that we use are the leading order in slow roll. The background evolution is solved however exactly.

[7] If one includes the running of the scalar and tensor spectral index then there are six observables.

$k_* = 0.05\,\mathrm{Mpc}^{-1}$ as,

$$n_s = 0.9665 \pm 0.0038 \quad (68\%\,\mathrm{CL})\,. \tag{3.32}$$

In slow roll approximation, n_s can be expressed in terms of slow roll parameters ϵ_1 and $\epsilon_2 = (d/dN)\ln(\epsilon_1)$ as,

$$n_s = 1 - 2\epsilon_1 - \epsilon_2\,, \tag{3.33}$$

where higher order (quadratic, *etc.*) corrections in slow roll parameters have been neglected. On the other hand, we have no measurements of the tensor spectrum. The current upper bound is expressed in terms of the scalar-to-tensor ratio, defined as,

$$r = \frac{\Delta_s^2}{\Delta_t^2}\,, \tag{3.34}$$

which to leading order in slow roll parameters reads, $r = 16\epsilon_1$, and it is constrained by the data at $k_* = 0.002\,\mathrm{Mpc}^{-1}$ as,

$$r < 0.065 \quad (95\%\,\mathrm{CL}) \tag{3.35}$$

Because r has not yet been measured, there is no meaningful constraint on the tensor spectral index, $n_t = -2\epsilon_1 = -r/8$. Sometimes one also quotes limits on the running of the scalar spectral index, defined as $\alpha_s = dn_s/d\ln(k)$, which makes up the fifth Gaussian observable. The current constraints on α_s are quite modest, $-0.013 < \alpha_s < 0.002$, and they are not strong enough to significantly constrain our model, in which α_s is second order in slow roll parameters and thus quite small.

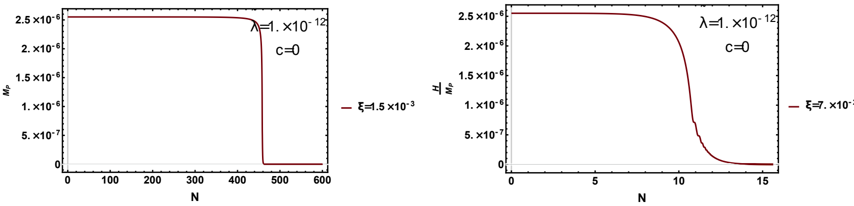
In what follows, we investigate how the model predictions depend on the parameters of the model ξ, λ and α. The dependence on α is the simplest, as it is completely fixed by the COBE normalization and by the scalar-to-tensor ratio r as,

$$\alpha = \frac{1}{24\pi^2 r A_{s*}}\,. \tag{3.36}$$

Now, as we shall see below, r can be well approximated by, $r \approx 0.003$ (with $\sim 30\%$ accuracy), implying that $\alpha \approx 7 \times 10^8$.

Next we look at the dependence on ξ. In figure 3.4 we illustrate how the duration of inflation depends on ξ. The case $\xi \ll 1$ falls into the class of *large field models* (left panel) and one gets $N_{\rm tot} \gg 60$. On the other hand, the case $\xi \lesssim 1$ is a *small field model* and yields only a few e-folds, *i.e.* $N_{\rm tot} \ll 60$ (right panel), making it not viable for inflationary model building. We note that one gets enough e-foldings of inflation ($N_{\rm tot} \sim 60$) when $\psi_m \sim 3 - 4$, which belongs to the class of *intermediate field models*, in which during inflation the inflaton ψ rolls over approximately one (unreduced) Planck mass, $\Delta\psi \approx \psi_m \sim m_{\rm P} = \sqrt{8\pi}M_{\rm P}$.

In figure 3.5 we show how n_s and r change as ξ varies for a fixed $\tilde\lambda \equiv \alpha\lambda$. The figure shows that, in the limit when ξ is very small, one reproduces the n_s and r of Starobinsky's model; increasing ξ introduces a deviation from



(A) For $\xi = 1.5 \cdot 10^{-3}$ inflation lasts for long time, much more than the required 60 e-foldings.

(B) For a largish ξ ($\xi = 0.07$) inflation does not any more last long enough as $N_{\text{tot}} \simeq 10 \ll 60$.

FIGURE 3.4: The duration of inflation in a large (left) and small (right) field inflationary model.

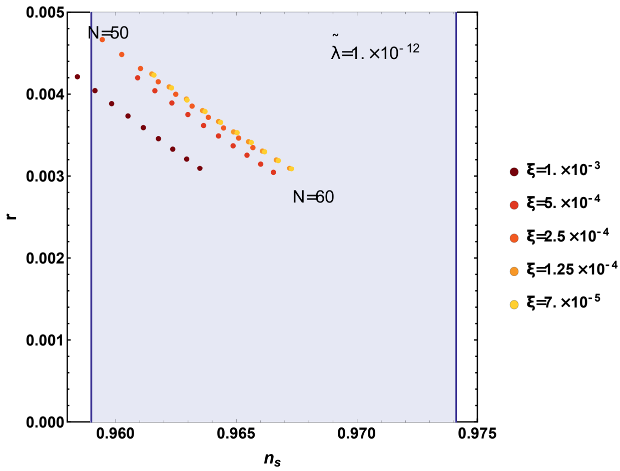


FIGURE 3.5: r *vs* n_s for varying ξ. As ξ decreases, r and n_s approach those of Starobinsky's inflation. As ξ increases, n_s and r mildly decrease.

Starobinsky's model such that both n_s and r decrease. Note that ξ cannot be too large, since n_s decreases as ξ increases, eventually dropping outside the region of validity of figure 3.5. In figure 3.7, we plot the running of the scalar spectral index, α_s, and the corresponding observational 2σ contours. As one can see, our model predicts a rather small α_s, which is well within the experimental limits when $\xi \ll 1$.

In figure 3.6 we show the effect of the hyperbolic field space geometry on the tensor to scalar ratio. Compared to the case of flat geometry, the slow roll parameters stay smaller for a longer time (in this sense the hyperbolic potential is more "flat"), which eventually translates into a suppression in the tensor to scalar ratio. For small values of $\xi \simeq 10^{-4}$ the flat space model is excluded, while the hyperbolic space model gives reasonable predictions as long as the condition $\xi^2 \gg 16\alpha\lambda$, as described in section 3.3, is satisfied.

Finally, let us discuss the dependence on λ. As explained above, λ primarily controls the vacuum energy at the end of inflation, and for that reason it

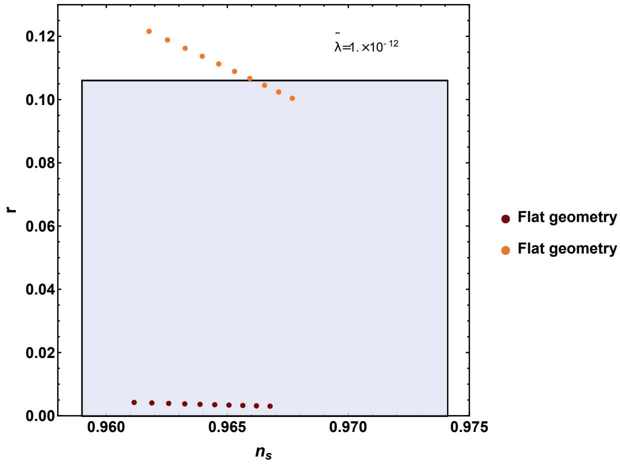


FIGURE 3.6: r *vs* n_s for $\xi = 0.25 \times 10^{-3}$ in the two cases of hyperbolic and flat field space geometry. Here the box indicates the 2σ contour as given in [7]. As we can see the flat geometry exceeds the 95% confidence level contours for r for this particular value of ξ, while the hyperbolic geometry sits comfortably at a low value of r.

ought to be small enough. When $\alpha\lambda \ll \xi^2/16$ the inflationary observables depend only very weakly on λ, and the dependence on λ starts becoming significant as $\alpha\lambda \sim \xi^2/16$ or larger. However, when the latter condition is satisfied, the vacuum energy left when ψ settles on its minimum is big enough to quickly dominate the universe and drive a second phase of accelerated expansion. In this case, the model would predict eternal inflation and would be as such ruled out. Hence we must require $\lambda \ll \xi^2/16$, in which limit any dependence on λ essentially drops out.

3.4.2 The role of the flat direction

So far we have not yet discussed how the model predictions depend on the dynamics of the flat direction χ. Unless the initial kinetic energy stored in χ is large, it will not affect the above analysis in any significant manner. Consider however the case when the initial kinetic energy in χ is large. This is equivalent to taking the parameter c defined in Eq. (3.28) to be initially large. One can easily convince oneself that in this case the energy density of the Universe will early on scale as, $\propto c^2/[a^6 \cosh^2(\psi)]$, which corresponds approximately to a period of kination, during which kinetic energy of a scalar field dominates and $\epsilon_1 \approx 3$. A brief period of kination followed by slow roll inflation is clearly visible in figure 3.8. From the value of the Hubble parameter at the beginning of inflation, $H_I \simeq 10^{13}$ GeV, we easily get an estimate

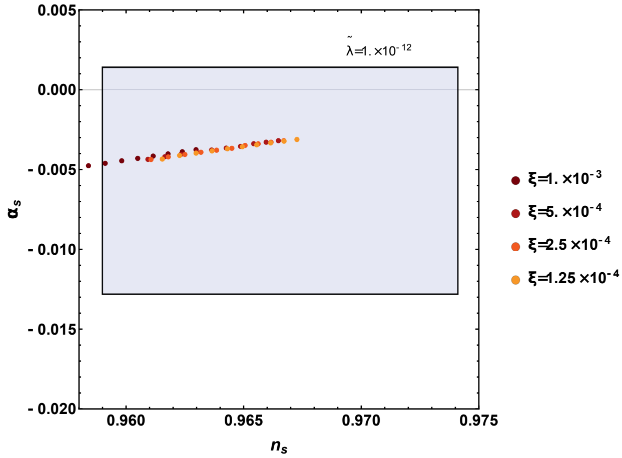


FIGURE 3.7: α_s *vs* n_s for varying ξ. As ξ decreases, α_s does not vary much, while n_s approaches the value of Starobinsky's inflation. The rectangle limits this time denote the 1σ contour as given in [7].

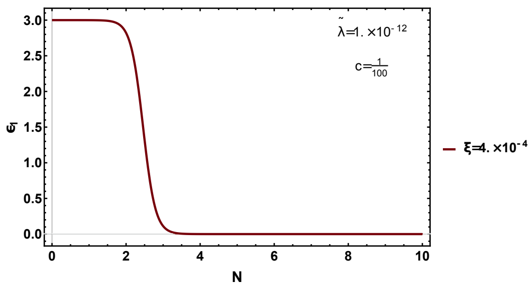


FIGURE 3.8: The parameter c introduces a brief period of kination, after which the universe transits to a quasi de Sitter epoch.

for the maximum number of e-foldings in kination,

$$(N_{\rm kin})_{\rm max} \lesssim \frac{1}{6} \ln \left(\frac{M_{\rm P}^2}{H_I^2} \right) \simeq 4\,, \tag{3.37}$$

where the estimate is obtained by assuming that the initial energy density in kination is at most Planckian.

A proper analysis of the spectrum of scalar cosmological perturbations requires solving for small perturbations of two fields, where the adiabatic mode is a linear combination of the two fields, fluctuations of χ and of ψ. During kination mostly χ will source scalar cosmological perturbations, during inflation it will be ψ, while at the transition from kination to inflation it will be a linear combination of the two. Therefore as a rough approximation one can assume: the fluctuations of the χ field source the adiabatic mode during kination, the fluctuations of the ψ field source it during inflation, and the transition period is instantaneous. [8] An inspection of figure 3.8 suggests that the transition lasts for about one e-folding, of equivalently about one Hubble time, $\Delta t_{\text{transition}} \sim 1/H$. This then means that sub-Hubble modes ($k/a \gg H$) behave adiabatically, *i.e.* their matching leads to an exponentially suppressed mixing between positive and negative frequency modes, and thus to an exponentially suppressed particle production and the amplification for these modes can be neglected. On the other hand, super-Hubble modes ($k/a \ll H$) can be treated in the sudden matching approximation, such that these modes inherit the highly blue spectrum from kination, $\Delta_s \propto (k/k_*)^3$, for which $n_s \simeq 4$. The resulting power spectrum is shown in figure 3.9.

As it was originally pointed out by Starobinsky [79] (see also [80]), apart from a *break* in its slope, the scalar power spectrum also exhibits a *memory* effect, manifested as *damped oscillations* which propagate into quite large momenta. These oscillations can be clearly seen in figure 3.9, where for definiteness we have assumed that the Hubble parameter at the matching equals to $k \sim 3H_0$ (which roughly corresponds to the CMB multipole $l \sim 3$). The scale at which the spectrum breaks can be shifted left or right by a suitable change in the Hubble parameter at the matching, or equivalently by changing the duration of the inflationary phase. Even though the oscillations are generated by the matching at $k \sim 3H_0$, they are visible all the way to $k \sim 30H_0$, or equivalently $l \sim 30$. In order to better understand whether these oscillations are strong enough to be detectable we show the upper and lower contours obtained by adding and subtracting the cosmic variance (shown as green lines). We emphasize that, even though the size of the oscillations is smaller than the cosmic variance, their cumulative effect might be statistically significant and one should look for their effect in the data.

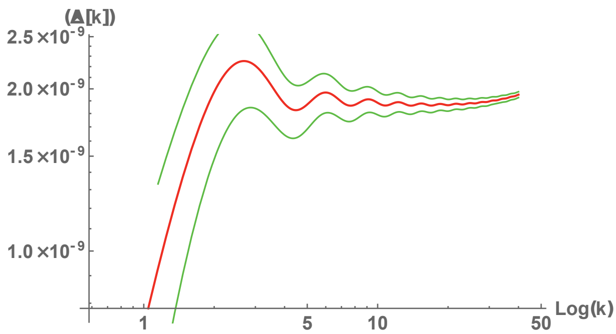


FIGURE 3.9: Power Spectrum in the sudden matching approximation. Notice the sudden drop in power on the largest scales. The upper and lower green lines represent the cosmic variance contours.

As it is known from the Planck Collaboration [7] (see also the earlier observations of COBE and WMAP [6, 81]), the temperature anisotropy spectrum of CMB has a dip in the low l part of the spectrum which has statistical

[8] This approximation neglects the effect of turning of the trajectory in field space [76], and a proper understanding of this effect we postpone for a future publication.

significance of $2-3\sigma$ which could be explained by a pre-inflationary period of kination. Such an explanation is viable only if inflation lasts for about 60 e-foldings. While this may seem like a tuning of inflation, we point out that 60 e-foldings are obtained in our model if the field rolls during inflation by about 1 (unreduced) Planck scale, which might be the natural scale of inflation. Namely, large field inflationary models might be hampered by quantum operators of large dimensions. In our model however, such operators may be forbidden by conformal symmetry; for a recent discussion of the role of such operators for inflation see [82].

3.5 Quantum corrections at 1-loop

In this section we investigate the quantum corrections to the action (3.6) and show that the model, and in particular the hierarchy required for the inflationary observables to match the Planck data, *i.e.* $\lambda \ll \xi \ll \alpha$, is preserved by quantum corrections.

So far we have learned that the theory (3.9) contains one massive scalar field ϕ (defined in Eq. (3.12)), $\mathcal{N}-1$ massless Goldstone bosons, one flat direction (denoted by χ), one conformal gauge degree of freedom (denotes by ω), one vector degree of freedom (which can be removed by setting $\sigma = 0$) and one massless, spin two degree of freedom (the graviton). For simplicity here we shall consider the O(1) case, in which there are no Golstone bosons. Furthermore, the contribution of the massless scalar χ to the (infrared) effective action is suppressed when compared to that of the massive scalar and therefore can be neglected.

Let us now consider the one loop contribution of the massive scalar ϕ. As it is well known, the quantum effective action that describes the quantum theory is given, at one loop, by the classical action plus the term

$$(i/2)\mathrm{Tr}\log(i\Delta)^{-1}\,,$$

where $i\Delta$ is the scalar propagator of the theory, which for the action (3.6) is given by the solution of,

$$\left[\nabla_\mu\nabla^\mu + \xi R - 3\lambda\phi^2\right] i\Delta(x;y) = i\frac{\delta^D(x-y)}{\sqrt{-g(x)}}\,, \tag{3.38}$$

where ϕ is the value of the (possibly space-time dependent) scalar field condensate, *i.e.* $\phi = \langle\hat{\phi}\rangle$ and D is the dimension of space-time.

While the effective action is highly non-local [83], its infrared limit (in which $\xi R, \lambda\phi^2 \gg \|\Box\|$) is quite simple. Indeed, as we calculate explicitely in chapter 5, the unregularized effective action is given by,

$$\begin{aligned}\Gamma &= \int \frac{d^D x}{(4\pi)^{D/2}} \sqrt{-g} \Bigg[\left(-\left(\xi + \frac{1}{6}\right) R + 3\lambda\phi^2 \right)^{\frac{D}{2}} \Gamma\left(-\frac{D}{2}\right) \\ &- \frac{1}{3}\left(\frac{1}{180} R_{\alpha(\beta\gamma)\delta} R^{\alpha(\beta\gamma)\delta} - \frac{1}{180} R_{(\alpha\beta)} R^{(\alpha\beta)} + \frac{(D-2)^2}{48} \mathcal{T}_{\alpha\beta} \mathcal{T}^{\alpha\beta} \right) \\ &\times \left(-\left(\xi + \frac{1}{6}\right) R + 3\lambda\phi^2 \right)^{\frac{D-4}{2}} \Gamma\left(2 - \frac{D}{2}\right) \Bigg] .\end{aligned} \tag{3.39}$$

From this action we can read off the quantum corrections to the interaction vertices. We find,

$$\delta\alpha = \frac{\left(\xi + \frac{1}{6}\right)^2}{16\pi^2} \log\left[\frac{3\lambda\phi^2 - \left(\xi + \frac{1}{6}\right) R}{\mu^2} \right] , \tag{3.40}$$

$$\delta\xi = -\frac{\left(\xi + \frac{1}{6}\right)\lambda}{8\pi^2} \log\left[\frac{3\lambda\phi^2 - \left(\xi + \frac{1}{6}\right) R}{\mu^2} \right] , \tag{3.41}$$

$$\delta\lambda = -\frac{\lambda^2}{16\pi^2} \log\left[\frac{3\lambda\phi^2 - \left(\xi + \frac{1}{6}\right) R}{\mu^2} \right] , \tag{3.42}$$

where we assumed that, $-\left(\xi + \frac{1}{6}\right) R + 3\lambda\phi^2 > 0$. This shows that the quantum corrections at one loop can maintain a specific hierarchy between the coupling constants, namely,

$$\lambda \ll \xi \ll \alpha\,, \implies \delta\lambda \ll \delta\xi \ll \delta\alpha\,, \tag{3.43}$$

which also happens to be the required hierarchy to obtain convincing inflationary predictions, as we showed in the previous section.

This discussion, however, does not take into account possible quantum gravitational corrections. While to reliably estimate their contribution one should set up a perturbative quantum gravity calculation, such as the one that has performed in [59], we also expect these contribution to yield corrections that are suppressed at energy scales below the Planck energy. As long as the space-time curvature remains within such a bound, therefore, we do not expect quantum gravitational correction to substantially change the conclusions of this section.

3.6 Remainder cosmological constant and naturalness

As we have seen in section 3.4, the parameter λ in the model (3.20) controls the vacuum energy left once inflation end. Such a vacuum energy will contribute to the cosmological constant, which is observed to be unexplicably small. In [74] we argued that Weyl symmetry provides a naturalness argument to explain why the cosmological constant is so small. In the following we report such discussion.

In 1979 ’t Hooft [84] proposed an explanation to why a physical parameter may be small. This *technical naturalness hypothesis* states that:

> *A physical parameter $\alpha(\mu)$ [..] is allowed to be very small only if the replacement $\alpha(\mu) \to 0$ would increase the symmetry of the system.*

’t Hooft then observes that the smallness of the parameter is protected in the sense that – due to the enhanced symmetry – quantum corrections will necessarily be proportional to $\alpha(\mu)$ – and thus will not affect the smallness of the parameter, explaining the term ‘technical’.

In what follows we argue that, once a small cosmological constant is generated through radiative breaking of conformal symmetry [1, 23, 31], it is protected from growing large (*cf.* Ref. [85]).

Local Weyl symmetry is an internal symmetry that – in addition to diffemorphism invariance – naturally survives a breaking of local conformal symmetry which we assume to be realised at very high energies/short distances. As we saw, because space-time torsion changes lengths of parallelly transported tensors, torsion tensor is the natural candidate which imbues Weyl symmetry in the gravitational sector [72][9], the symmetry transformations defined in equations (3.2–3.3–3.4).

These transformations then imply that classical gravity in vacuum is conformal. It is straightforward to extend the symmetry (3.2) to the matter sector [72], as we saw in chapter 2. The coupling of gravity to matter can then be made conformal by adding a dilaton field (whose condensate determines the value of the Newton ‘constant’). Finally, quantum effects break conformal symmetry [23, 86] and in what follows we discuss in which way these breakings affect the observed cosmological constant.

The *cosmological constant problem* (CCP) is by far the most severe hierarchy problem of physics, and up to date no convincing solution has been proposed that is accepted by most physicists. Assuming the observed cosmological constant (CC) is given by dark energy then (in dimensionless units): $(8\pi G_N)\Lambda \sim 10^{-122}$. The CCP can be stated as follows [87] (for reviews see also [88–90]):

1) Why is the cosmological constant so small (when measured in natural units)?

[9] Very much like the longitudinal component of the vector field in an Abelian gauge theory, the longitudinal component of the torsion trace vector contains the compensating scalar that implements Weyl symmetry to Einstein’s vacuum equations.

2) Why is it becoming important right now (when we are observing), i.e. why is the energy density in CC so close to the energy density in matter fields, $\rho_\Lambda \equiv \Lambda/(8\pi G_N) \sim \rho_m$?

3) If CC is diffferent from zero, what sets its magnitude and what stabilizes it against running with the energy scale?

To elaborate on *Problem 1*, note that quantum vacuum fluctuations contribute to ρ_Λ as $\sim k_{\rm UV}^4$, where $k_{\rm UV}$ is an ultraviolet momentum cutoff scale. Given that the natural cutoff of quantum gravity is the Planck scale (the scale at which gravity becomes strongly interacting), $m_{\rm P} = 1/\sqrt{G_N}$, the first problem can be rephrased as: *Why is* $\Lambda/m_{\rm P}^2 \ll 1$*?* In other words, why vacuum fluctuations do not (significantly) contribute to Λ? As regards *Problem 3*, we note that, if one can identify the symmetry which is realised when CC vanishes, then this symmetry protects CC from running fast with scale.

If a theory of gravity, matter and interactions is classically conformal, still quantum effects can violate the classical symmetry. The couplings constants can start running in such a way that the potential develops a new minimum, away from the origin of the field space thus introducing an energy scale and breaking conformal symmetry [1]. If the couplings are small at some fiducial large energy scale μ_*, their running will typically be slow, allowing for a large hierarchy between the UV scale and the scale of symmetry breaking. The latter can be estimated as the scale at which a given coupling turns negative, allowing for a minimum to form. From the perturbative treatment of the running we obtain an estimate on that scale, $\mu \sim \mu_* \exp(-1/\lambda_*)$, where λ_* denotes the relevant coupling at the scale μ_* and we have dropped factors of $\mathcal{O}(1)$ in the exponent. Assuming that the vacuum expectation value of the scalar field is of the order of the scale μ we obtain a rough estimate

$$\rho_\Lambda \sim -v^4 \sim -\mu^4 \sim -\mu_*^4 \exp(-\frac{4}{\lambda_*}). \tag{3.44}$$

In light of the above and with the right choice of the couplings at μ_* ($\lambda_* \sim 10^{-2}$), one could, in principle, get a cosmological constant as small as the observed one (though negative). In practice, however, Nature has chosen to break conformal symmetry in the matter sector at the electroweak scale, $\mu \sim 10^2$ GeV, at which the Higgs field acquires an expectation value of $\langle h \rangle \equiv v \simeq 246$ GeV, which is responsible for the mass generation of all standard model particles (except perhaps of the neutrinos).[10] This then sets the natural energy density scale for the cosmological constant,

$$\rho_\Lambda^{\rm EW} \sim -v^4 \sim -10^8 \text{ GeV}. \tag{3.45}$$

[10] In the case of conformally symmetric theory the BEH mechanism of the standard model is replaced by the Coleman–Weinberg mechanism [1].

The contribution (3.45) must be negative, since in the absence of Higgs condensate $\rho_\Lambda^{\rm EW}$ must vanish.[11]

It seems that we have a *no-go* theorem: The contribution from matter field condensates is necessarily large and negative, while the observed cosmological constant is small and positive. In order to overcome this impasse, we ought to dig deeper into the model and understand how gravity contributes to the cosmological constant. To get a clearer picture, let us consider again the model (3.6), consisting of the metric field $g_{\mu\nu}$, the torsion field $T^\alpha_{\ \rho\sigma}$, a singlet dilaton ϕ and matter fields ψ_i. The action is given by [72],

$$\begin{aligned} S[\phi, g_{\mu\nu}, T^\alpha_{\ \rho\sigma}, \psi_i] = \int \sqrt{-g} d^4x \Big\{ \frac{\xi}{2}\phi^2 R + \alpha R^2 - \frac{1}{2} g^{\mu\nu} \overline{\nabla}_\mu \phi \overline{\nabla}_\nu \phi \\ - \frac{\lambda}{4}\phi^4 \Big\} + S_m[\psi_i, g_{\mu\nu}, T^\alpha_{\ \rho\sigma}] \,, \end{aligned} \tag{3.46}$$

where and S_m denotes the matter action.

This action can be discerning for inflation [91], but it can be also used to get an insight on how the gravitational sector contributes to the cosmological constant. The action (3.46) contains three scalars: ϕ, R (scalaron) and the longitudinal component of torsion trace,

$$\mathcal{T}^L_\mu = \partial_\mu \phi^0(x) \,. \tag{3.47}$$

In the absence of scalar condensates, the theory (3.46) is at its conformal fixed point, and the vacuum energy must vanish. In order to understand what happens away from the conformal point, it is instructive to go to the Einstein gauge, as we described in section 3.2. The procedure turns (3.46) into the (on-shell) equivalent action for the gravitational sector (3.9),

$$\begin{aligned} S_g[\phi, g_{\mu\nu}, T^\alpha_{\ \rho\sigma}] = \int \sqrt{-g} d^4x \Big\{ \frac{\omega^2}{2} R - \frac{1}{2} g^{\mu\nu} \overline{\nabla}_\mu \phi \overline{\nabla}_\nu \phi \\ - \frac{1}{16\alpha}\left(\xi\phi^2 - \omega^2\right)^2 - \frac{\lambda}{4}\phi^4 \Big\} \,. \end{aligned} \tag{3.48}$$

This (Einstein frame) action is (classically) conformal only if the Lagrange multiplier field transforms as, $\omega \to \Omega^{-1}\omega$ and it reduces to the usual general relativity coupled to a real scalar in a gauge in which ω is gauge fixed to the (reduced) Planck mass,

$$\omega = M_{\rm P} \qquad ({\rm gauge\ \ fixing}) \,, \tag{3.49}$$

with $M_{\rm P} \equiv 1/\sqrt{8\pi G_N}$ (any choice for ω is in principle allowed). Since $M_{\rm P}$ is the only scale in the problem, it has no absolute meaning, *i.e.* conformal symmetry of (3.48) teaches us that the choice (3.49) is *physically equivalent* to any other non-vanishing (local) scale $\omega'(x) = \Omega^{-1}(x) M_{\rm P}$. The two remaining scalars, ϕ^0 and ϕ, are the physical (scalar) degrees of freedom of the theory.

[11] Another negative contribution is generated by the chiral condensate of mesons generated as the chiral symmetry of QCD gets broken by the chiral anomaly but – when compared with (3.45) – that contribution can be neglected since it is of the order -10^{-4} GeV.

Since ϕ^0 stems from Weyl symmetry, ϕ^0 retains a flat direction, *i.e.* it exhibits a (global) shift symmetry, $\phi^0(x) \to \phi^0(x) + const$ and thus cannot contribute to the cosmological constant. On the other hand, ϕ exhibits a nontrivial potential. A simple calculation shows that, when $\alpha/\xi > 0$, ϕ is tachyonic and condenses to, $\phi_0^2 = \xi\omega^2/(\xi^2 + 4\lambda\alpha)$, at which the mass and potential energy are given by,

$$m(\phi_0)^2 = \frac{\xi}{2\alpha}\omega^2\,, \qquad V(\phi_0) = \frac{\lambda}{4(\xi^2 + 4\lambda\alpha)}\omega^4\,. \tag{3.50}$$

Let us pause to try to understand the result (3.50). As a consequence of conformal symmetry breaking, the gravitational sector produces a positive cosmological constant whose size in natural (dimensionless) units is given by,

$$\frac{V(\phi_0)}{\omega^4} = \frac{\lambda}{4(\xi^2 + 4\lambda\alpha)}\,. \tag{3.51}$$

For this to compensate the negative cosmological constant generated in the matter sector, one ought to fine tune (3.51) to be $\sim 10^{-65}$, such that when (3.51) is added to the matter contribution (3.45), one obtains the observed cosmological constant, $\Lambda/\omega^2 \sim 10^{-122}$. We emphasize that, once the cosmological constant is tuned to the observed value, 't Hooft's technical naturalness ensures that quantum corrections (both from gravitational and matter fields) will not affect it. This can be made more precise as follows. Let us assume that $V_{\rm eff} \sim 10^{-122}\omega^4$ represents the total contribution to the cosmological constant. Then, the RG improved $V_{\rm eff}$ must obey the Callan-Symanzik equation,

$$\mu\frac{d}{d\mu}V_{\rm eff}(\phi, \text{all other fields}) = \mu\frac{d}{d\mu}V_{\rm eff}(0, \text{all other fields} \to \text{fixed point})\,, \tag{3.52}$$

where the second term constitutes $V_{\rm eff}$ with all fields set to their respective conformal fixed point (at which all scalars vanish), which vanishes for the conformal theory under study.[12] Eq. (3.52) tells us then that $V_{\rm eff}$ (and therefore also Λ) does not change if the scale μ changes. This means that, while the precise value of the fields and couplings can depend on μ, the value of the effective potential at its minimum cannot.[13]

To conclude this section, let us recall that the classical conformal symmetry alleviates the hierarchy problem associated with the mass of the Higgs boson present in the standard model [93]. In this way imposing conformal symmetry on physical theories can elucidate the most notorious hierarchy problems in physics – the cosmological constant problem and the gauge hierarchy problem.

[12] If the potential is nonzero at the origin of the field space, one has to cancel the zero-point energy order by order to obtain a homogenous RG equation for the effective potential [92].

[13] In practical applications $V_{\rm eff}$ is always approximated by its truncated version at a finite order in loop expansion. Such a truncated effective potential contains some residual dependence on μ [12], which is however suppressed by a suitable power of $\hbar$.

3.7 Conclusions

We investigate a simple inflationary model (3.6) which exhibits local Weyl (or conformal) symmetry at the classical level which is realized by the space-time torsion. We show that in its simplest realization the model contains two scalar fields and one massless spin two field. One of the scalars corresponds to a flat direction χ, which is a remnant of conformal symmetry and therefore the flatness is protected from quantum corrections. The other scalar (ψ) is massive and can support inflation. The noteworthy property of the model is a negative configuration space curvature[14] which is responsible for the flattening of the effective potential for ψ which is crucial for obtaining a long lasting inflation. While this feature can be present in other realisation of the spontaneous symmetry breaking pattern such as in the model of [56], it appears that the maximal symmetry is related to the enhanced, local, Weyl symmetry. The predictions of low tensor to scalar ratio, which seems typical for Weyl invariant models, seems to be in line with the results found in [29]. Our analysis shows that, for a typical choice of parameters of the model, the model approximately reproduces the results of Starobinsky inflation, which again seem to appear as a 'universal attractor'. However, variation of the coupling constants yields significant variation of in the predictions of the model as regards the scalar spectral index n_s and the tensor-to-scalar ratio r, which can be used to test the model by the next generation of CMB satellite missions such as COrE [94]. Therefore, for quite a large range of couplings (namely $\xi < 0.01$, $\lambda \ll \xi$), our model is viable in light of the currently available data. We also point out that, if a lot of energy is initially stored in the flat direction, the universe will undergo a short period of kination, followed by quasi-de Sitter inflation. Such a sequence is characterized by a lack of power in low momentum modes, a break in the power spectrum and a memory manifested as damped oscillations in the power spectrum.

[14] A similar feature flanks the so-called α-attractor models [52] constructed from a supergravity model.

Chapter 4

Observing geometrical torsion

4.1 Introduction

The recent developments in observing gravitational waves (*e.g.* [95]) are an exciting development from high energy physics perspective. This is because they might offer a window on new physics, that might shed light on the puzzles theoretical physicists are facing, concerning the behaviour of gravity in the strong field regime. Several observables carry information about the behaviour of gravity at short distances, such as the vibrational modes of the black hole final state, and can be used in principle to test modified gravity theories and, perhaps, carry information about the quantum behaviour of gravity.

Alas this information is not easily accessible. The reason being the smallness of the signal, when compared to background noise from other sources. To overcome this difficulty, the strategy has been to know beforehand what to look for in the datas. This is done by analytical and numerical tools, used to calculate the spectrum of gravitational waves emission, the so-called wave forms templates. This is then compared to the data taken by three detectors, and a detection is claimed if all of them agree.

As one can imagine, this approach has some flaws, namely, if we want to see anything different than expected, we would have to produce different templates than the ones used in general relativity, and perform a statistical study. Arguably, when the space based interferometer, LISA, comes online, we will be able to observe many cycles of emission, not just the final burst currently accessible by LIGO and VIRGO. This will allow more precise comparison of the expected template, with the actual signal, and might reveal more subtle differences, for example in the *phase* of the gravitational wave.

With this goal in mind, it becomes important to study precisely modified theories of gravity, to understand in which way they might induce a different signal in the upcoming observations. In this chapter we attempt to answer this question in the context of gravity with torsion. We will argue first that torsion would appear as a signal in gravitational waves detectors as they are constructed now, by using the Jacobi equation (2.27). We will then focus on the torsion trace, and in particular to the dilaton mode, which is supposed to be massless if scale symmetry is broken spontaneously, and therefore propagates at large distances.

Cartan-Einstein (CE) theory is a very old topic (for reviews that are still actual see [18, 36]), and yet as far as we are aware of there has been no proposal to detect (dynamical) torsion *via* instruments such as gravitational wave detectors. In this chapter we argue that conventional gravitational wave detectors can be used to detect propagating (dynamical) torsion. The probable reasons why this has not been proposed before are (a) skew symmetric torsion (which is typically what one means by the torsion in Cartan-Einstein theory) does not imprint any signal on gravitational wave detectors, (b) torsion is not dynamical in Cartan-Einstein theory and, most notably, (c) we have no understanding of torsion waves sources, which is needed to construct waveforms used to detect gravitational waves above the foreground noise.

Furthermore, in the original CE theory torsion figures as a constraint (non-dynamical) field, which exists locally where the source is, but does not propagate. However, when matter is quantized, one can show that (when one integrates out matter fields) already at the one-loop level, both torsion trace and skew symmetric torsion become dynamical [18, 42], such that they can propagate through space in form of torsion waves and carry energy and information throughout our universe just as gravitational waves do. We do not know whether this is realized in nature, but if true the possibilities are exciting enough to deserve a closer attention of both theorists and observers.

If Weyl symmetry is realized at the classical level then torsion trace vector couples to scalar fields, implying that scalars source torsion trace [72], modifying thus the original Cartan-Einstein theory. Even in this case, the theory would be difficult to test, since the torsion trace vector at tree level *only* couples to scalars, *i.e.* not to gauge fields or fermions, it will become hidden at solar system's scales [96]. However, if torsion is dynamical, and if the scale symmetry is spontaneously broken by a Coleman-Weinberg mechanism, we expect the existence of a dilaton field, a massless goldstone mode of the broken scale symmetry, which is sourced by strong space-time curvature condensates, and by the trace anomaly, and propagates at long distances.

Recent developments [25] also show that, for such a dilaton, which we identify as scalar torsion trace, one can write down rather generic equations of motion and propagation, which couple said scalar to the anomalous trace of the energy momentum tensor. These constitute the basis for performing a post newtonian study and numerically produce waveform templates that are then to be compared with data. Since tensors and scalars decouple at linear order, the corrected wave forms will be similar to those constructed in general relativity, but at higher orders in the post newtonian expansion they will appear.

Torsion has been used in literature for various purposes. For example, torsion was proposed as a way of avoiding the Big Bang singularity [42, 97, 98], to drive inflation [99] and create perturbations [100] or primordial magnetic fields [101], to generate dark matter [102] or dark energy [103, 104]. Apart from being detectable by gravitational wave detectors, torsion might be also detectable at the LHC [105].

4.2 Jacobi Equation

Just like in general relativity, where gravitational waves induce a change in distance between two test bodies described by the Jacobi equation, in a more general geometric theory that contains geometric torsion the suitably generalized Jacobi equation [72] governs the distance between two test bodies. In Chater 2 we have shown that the Jacobi equation reads,

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J + 2\nabla_{\dot{\gamma}}T(\dot{\gamma}, J) = R(\dot{\gamma}, J)\dot{\gamma}\,, \tag{4.1}$$

where $T(\cdot,\cdot)$ and $R(\cdot,\cdot)$ denote the torsion and curvature tensors, respectively, and $\nabla_{\dot{\gamma}}$ is the covariant derivative in the direction of the tangent vector $\dot{\gamma}$. Usually Jacobi vector fields J are taken to be orthogonal to $\dot{\gamma}$ ($g(J,\dot{\gamma}) = 0$, where $g(\cdot,\cdot)$ is the metric tensor), but that in fact is not necessary since the solution for J in the direction of $\dot{\gamma}$ are trivial. Let us first derive how torsion enters Eq. (4.1).

Jacobi field J represents a space-time vector field that characterizes the distance between neighboring geodesics in a congruence of geodesics and it is therefore useful in determining how gravitational wave detectors respond to a passing (gravitational or torsion) wave.

In a space-time with geometric torsion, metric compatibility condition,

$$\nabla_\mu g_{\alpha\beta} = 0\,,$$

implies that the connection associated with the covariant derivative ∇ can be written as, $\Gamma^\alpha{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} + K^\alpha{}_{\mu\nu}$, where $\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu}$ is the Christoffel connection (Levi-Civita symbol) and $K^\alpha_{\mu\nu}$ denotes the contorsion tensor defined as,

$$K^\alpha{}_{\mu\nu} = T^\alpha{}_{\mu\nu} + T_{\mu\nu}{}^\alpha + T_{\nu\mu}{}^\alpha\,,\ T^\alpha_{\mu\nu} = \Gamma^\alpha_{[\mu\nu]}\,. \tag{4.2}$$

The curvature tensor $R(\cdot,\cdot)$ in (4.1) can be conveniently written in terms of the Riemann curvature tensor,

$$\overset{\circ}{R}{}^\alpha{}_{\mu\rho\nu} = \partial_\rho\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\nu} - \partial_\nu\overset{\circ}{\Gamma}{}^\alpha{}_{\mu\rho} + \overset{\circ}{\Gamma}{}^\alpha{}_{\sigma\rho}\overset{\circ}{\Gamma}{}^\sigma{}_{\mu\nu} - \overset{\circ}{\Gamma}{}^\alpha{}_{\sigma\nu}\overset{\circ}{\Gamma}{}^\sigma{}_{\mu\rho}\,, \tag{4.3}$$

and the contorsion tensor K as,

$$R^\alpha{}_{\mu\rho\nu} = \overset{\circ}{R}{}^\alpha{}_{\mu\rho\nu} + 2\overset{\circ}{\nabla}_{[\rho}K^\alpha{}_{\mu|\nu]} + 2K^\alpha{}_{\sigma[\rho|}K^\sigma{}_{\mu|\nu]}\,, \tag{4.4}$$

where $\overset{\circ}{\nabla}$ represents the (general relativistic) covariant derivative taken with respect to the Christoffel connection $\overset{\circ}{\Gamma}$.

According to the Young classification - the torsion tensor T (anti symmetric in two indices by definition) can be decomposed as the sum of three distinct tensors,

1. torsion trace $\mathcal{T}_\mu = \frac{1}{3}T^\alpha{}_{\mu\alpha}$;
2. skew symmetric torsion $\Sigma_{\alpha\beta\gamma} = T^\mu{}_{[\beta\gamma}g_{\alpha]\mu}$;

3. mixed torsion tensor Q, obtained by subtracting the trace and skew symmetric parts from the torsion tensor, $T^{\lambda}{}_{\mu\nu}$.

Next, from (4.2) it follows that the contorsion tensor can be written as,

$$K_{\alpha\nu\gamma} = 2g_{\gamma[\nu}\mathcal{T}_{\alpha]} + \Sigma_{\alpha\nu\gamma} + Q_{\alpha\nu\gamma}\,, \tag{4.5}$$

from which we get,

$$\mathcal{T}_{\gamma} = \frac{1}{3}K^{\alpha}{}_{\alpha\gamma}\,, \qquad \Sigma_{\alpha\nu\gamma} = K_{[\alpha\nu\gamma]}\,, \tag{4.6}$$

$$Q_{\alpha\nu\gamma} = K_{\alpha\nu\gamma} - \frac{2}{3}g_{\gamma[\nu}\tau_{\alpha]} - \Sigma_{\alpha\nu\gamma}\,. \tag{4.7}$$

Making use of $\nabla_{\dot{\gamma}} = \dot{\gamma}^{\mu}\nabla_{\mu}$ the Jacobi equation (4.1) can be written in components as,

$$\begin{aligned}
\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}J^{\alpha} \;&=\; \mathring{R}^{\alpha}{}_{\mu\nu\rho}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}J^{\rho} + \dot{\gamma}^{\mu}\Big\{-\Big(\frac{\mathring{D}}{D\tau}K^{\alpha}{}_{\rho\mu}\Big)J^{\rho} \\
&-2\Big(\frac{\mathring{D}}{D\tau}T^{\alpha}{}_{\mu\rho}\Big)J^{\rho} + \Big(\frac{\mathring{D}}{D\tau}K^{\alpha}{}_{\mu\rho}\Big)J^{\rho} - \dot{\gamma}^{\nu}\Big(\mathring{\nabla}_{\rho}K^{\alpha}{}_{\mu\nu}\Big)J^{\rho}\Big\} \\
&+\dot{\gamma}^{\mu}\Big\{\dot{\gamma}^{\nu}K^{\rho}{}_{\nu\mu}\Big(\mathring{\nabla}_{\rho}J^{\alpha}\Big) - 2K^{\alpha}{}_{\rho\mu}\Big(\frac{\mathring{D}}{D\tau}J^{\rho}\Big) - 2T^{\alpha}{}_{\mu\rho}\Big(\frac{\mathring{D}}{D\tau}J^{\rho}\Big)\Big\}\,,
\end{aligned} \tag{4.8}$$

where we have neglected the terms quadratic in contorsion tensor and we have defined the general relativistic covariant derivative along a geodesic γ parametrized by an affine parameter τ as, $\mathring{D}/D\tau \equiv \dot{\gamma}^{\mu}\mathring{\nabla}_{\mu}$. When the torsion tensor decomposition as in (4.5) are inserted into (4.8) one obtains contributions from various components of the torsion tensor to acceleration of the Jacobi field along γ (again to leading order in torsion),

$$\begin{aligned}
\Big(\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}J^{\alpha}\Big)_{\mathcal{T}} \;&=\; \Big(\mathring{\nabla}_{J}\mathcal{T}^{\alpha}\Big) + \dot{\gamma}^{\alpha}\Big(\mathring{\nabla}_{J}\mathcal{T}_{\dot{\gamma}}\Big) \\
&-2\mathcal{T}^{\alpha}\Big(\frac{\mathring{D}}{D\mathcal{T}}J_{\dot{\gamma}}\Big) - \Big(\mathring{\nabla}_{\mathcal{T}}J^{\alpha}\Big) + \dot{\gamma}^{\alpha}\mathcal{T}_{\rho}\Big(\frac{\mathring{D}}{D\mathcal{T}}J^{\rho}\Big)\,,
\end{aligned} \tag{4.9}$$

$$\Big(\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}J^{\alpha}\Big)_{\Sigma} \;=\; 0\,, \tag{4.10}$$

$$\& \Big(\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\mathring{\nabla}_{\mu}\mathring{\nabla}_{\nu}J^{\alpha}\Big)_{Q} \;=\; -2\dot{\gamma}^{\mu}\Big(\frac{\mathring{D}}{D\mathcal{T}}Q^{\alpha}{}_{\mu\rho}\Big)J^{\rho}\,, \tag{4.11}$$

where $J_{\dot{\gamma}} = \dot{\gamma}^{\mu}J_{\mu}$, $\mathcal{T}_{\dot{\gamma}} = \dot{\gamma}^{\mu}\mathcal{T}_{\mu}$ and $\mathring{\nabla}_{J} = J^{\mu}\mathring{\nabla}_{\mu}$. We will now use these results to investigate how one can detect torsion waves induced by the torsion trace $\mathcal{T}$, skew symmetric torsion Σ or mixed torsion Q.

When this and Eq. (4.2) are used in (4.1), one obtains how different torsion components contribute to the acceleration of the Jacobi field (see Eqs. (4.9), (4.10) and (4.11). In what follows we analyze how different torsion components influence the distance J between neighboring geodesics.

4.3 Detection

In current earthly measurements (and planned measurements in space) any perturbations of spacetime can be viewed as a small perturbation away from Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Throughout this work we shall assume that both gravitational metric perturbations $h_{\mu\nu}(x) = g_{\mu\nu}(x) - \eta_{\mu\nu}$ and torsion perturbations $T^{\alpha}{}_{\mu\nu}(x)$ are small such that we can linearize in $h_{\mu\nu}$ and in $T^{\alpha}{}_{\mu\nu}$. In this linear approximation, to the required accuracy one can set, $\dot{\gamma}^{\mu} = (1,0,0,0)^T$.

Gravitational waves. To detect gravitational waves it is convenient to work in traceless, transverse (TT) gauge, in which $h_{\mu 0} = 0$, $\delta_{ij}h_{ij} = 0$ and $\partial_i h_{ij} = 0$. This is also known as the physical gauge because in this gauge h_{ij} is (gauge) invariant to linear coordinate shifts ξ^{μ}, *i.e.* $\mathcal{L}_{\xi} h_{ij} = 0$. From Eq. (4.8) we see that only $\overset{\circ}{R}{}^{i}{}_{00j}$ (and permutations of its indices) components contribute. Next, in TT gauge $\overset{\circ}{R}{}^{i}{}_{00j} = (1/2)\ddot{h}_{ij}(t,\vec{x})$ and $\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\overset{\circ}{\nabla}_{\mu}\overset{\circ}{\nabla}_{\nu}J^i$ can be approximated by dJ^i/dt^2 such that we have,

$$\frac{d^2 J^i}{dt^2} = \frac{1}{2}\ddot{h}_{ij}(t,\vec{x})J^j\,. \tag{4.12}$$

Gravitational waves are built from spin two massless gravitons, which come in two polarizations, known as the plus (+) and cross (×) polarization. For example, if the axes are chosen such that a plane gravitational wave is moving in the z-direction, then $h_{zz} = 0$ and,

A) Plus polarization: $h_{xx} = -h_{yy} = h_{+}\cos(\omega t - kz)$;

B) Cross polarization: $h_{xy} = h_{yx} = h_{\times}\cos(\omega t - kz)$;

where $\omega = ck$ ($k = \|\vec{k}\|$) is the frequency of the wave and h_{+} and $h_{\times}$ denote the amplitude of the $+$ and $\times-$polarized wave, respectively. By making the *Ansatz*, $J^i(t,\vec{x}) = J^i_{(0)} + \Delta J^i_{(0)}\cos(\omega t - kz)$, one can easily show that to leading order in h_{+} ($h_{\times}$) equation (4.12) is solved by,

A) Plus polarization: $J^x(t,z) = J^x_{(0)}\big[1 + (h_{+}/2)\cos(\omega t - kz)\big]$, $J^y(t,z) = J^y_{(0)}\big[1 - (h_{+}/2)\times\cos(\omega t - kz)\big]$;

B) Cross polarization: $J^x(t,z) = J^x_{(0)} + (h_{\times}/2)J^y_{(0)}\cos(\omega t - kz)\big]$, $J^y(t,z) = J^y_{(0)} + (h_{\times}/2)\times J^x_{(0)}\cos(\omega t - kz)\big]$.

These solutions show that the response displacements $\Delta J^i_{(0)}$ are in phase with the original wave and that for both polarizations the relative displacement:

$\Delta L/L = \Delta J^x_{(0)}/J^x_{(0)} = \Delta J^x_{(0)}/J^y_{(0)} = h_{+,\times}/2$ is given by one half of the wave amplitude.

Torsion trace. The contributions to the Jacobi field acceleration from the torsion trace vector is given by (4.9). Analogous to the contributions of Christoffel connection, the terms in the second line of (4.9) contribute at second order and thus can be neglected, such that we have,

$$\ddot{J}^0 = 0\,, \qquad \ddot{J}^i = J^0\dot{\tau}^i + J^j\partial_j\tau^i\,. \tag{4.13}$$

Now from $g(J,\dot{\gamma}) = \dot{\gamma}\cdot J = 0$ and $\dot{\gamma}^\mu = \delta^\mu{}_0$ it follows that (to this order) $J^0 = 0$ (which is consistent with the first equation in (4.13)) and the second equation in (4.13) simplifies to,

$$\ddot{J}^i = J^j\partial_j\tau^i\,. \tag{4.14}$$

To facilitate comparison with gravitational waves, we assume that τ^i can be written as a plane wave moving in the z-direction,

$$\tau^i = \tau^i_{(0)}\cos(\omega t - kz)\,. \tag{4.15}$$

The theory of torsion trace presented in Ref [72] can be compared to a gauge invariant Proca theory with a gauge field, τ_μ, that becomes massive after the conjectured symmetry breaking. For this reason, it is reasonable to assume that torsion trace waves propagate both longitudinal and transverse polarisations, for which $\tau^i_{(0)}$ are respectively given by,

$$\tau^i_{(0),L} = \delta^i_z\frac{\omega}{m}\,,\ \tau^0_{(0),L} = -\frac{\|\vec{k}\|}{m}\,, \tag{4.16}$$

$$\tau^i_{(0),T} = \frac{1}{\sqrt{2}}\left(\delta^i_x \pm \delta^i_y\right)\,,\ \tau^0_{(0),T} = 0\,. \tag{4.17}$$

Inserting the *Ansatz*,

$$J^i(t,z) = J^i_{(0)} + \Delta J^i_{(0)}\sin(\omega t - kz)\,, \tag{4.18}$$

into the Jacobi equation (4.14) and making use of (4.15) results in,

$$\Delta J^z_{(0)} = 0\,, \qquad \Delta J^{x,y}_{(0)} = -\frac{c^2 k}{\omega^2}\tau^{x,y}_{(0),T}J^z_{(0)} \approx -\frac{c}{m}\tau^{x,y}_{(0),T}J^z_{(0)}\,, \tag{4.19}$$

for the transverse polarisation, and, for the longitudinal polarisation,

$$\Delta J^z_{(0)} = -\frac{c^2 k}{\omega^2}\tau^z_{(0),L}J^z_{(0)} \approx -\frac{c}{\omega}\tau^z_{(0),L}J^z_{(0)}\,,\ \Delta J^{x,y}_{(0)} = 0\,. \tag{4.20}$$

Let us now pause to discuss this result. Notice firstly that the phase of the response function (4.19) is shifted by $\pi/2$ with respect to the phase of the original wave (this is to be contrasted with no phase shift in the case of gravitational waves). This phase shift may be difficult to observe, however, if we have some confidence that gravitational wave and torsion wave come from the same source (which can be established by having a directional information) and that dispersive effects due to the interstellar medium are negligible (a rough estimate indicates that the medium induces a phase shift on the gravitational wave which is about 10^{-22} for a source at the Hubble distance which can be safely neglected), then this phase shift might be observable. A second difference is geometric: while gravitational waves induce response in the relative length along the same transverse direction (for plus polarization) or along the opposite, but still transverse, direction (for cross polarization), torsion trace induces response along transverse direction which is proportional to the longitudinal direction of the instrument. Finally third (and probably the most important) difference between gravitational waves and torsion trace vector signature is in that the relative displacement (4.19) is inversely proportional to the frequency/wave vector (while no frequency dependence is present in the gravitational wave response). This difference may be crucial when distinguishing a torsion wave signature from that of a gravitational wave.

Skew symmetric torsion. From Eq. (4.10),

$$\ddot{J}^i = 0\,. \tag{4.21}$$

it immediately follows that skew symmetric torsion cannot be observed by gravitational wave detectors. [1]

Torsion with mixed symmetry. From Eq. (4.11) we see that,

$$\ddot{J}^i = -2\dot{Q}^i{}_{0j} J^j\,, \tag{4.22}$$

where we have assumed that $J^0 = 0$. Since we do not know any mechanism by which dynamical Q can be generated, [2] we cannot be sure how the wave equation for Q looks like. Therefore the analysis presented in what follows represents an educated guess. It is reasonable to assume that – just as any massless waves – the Q waves are transverse, motivating the following *Ansatz*,

$$Q^i{}_{0j} = Q^i{}_{j(0)} \cos(\omega t - kz)\,, \qquad \omega = ck \tag{4.23}$$

where $Q^i{}_{j(0)}$ is a symmetric 3×3 matrix and $Q^i{}_{z(0)} = 0 = Q^z{}_{j(0)}$ (transversality on both indices), such at it at most has 3 independent polarizations,

[1] This conclusion could have been reached by a careful look at the derivation of Jacobi equation (4.1) in Ref [72], where it is derived by making use of the geodesic equation, $\nabla_{\dot\gamma}\dot\gamma = 0$ and of equation $\mathcal{L}_{\dot\gamma} J = [J, \dot\gamma] = 0$. None of these equations contains any dependence on skew symmetric torsion.

[2] Namely, neither matter (scalars, fermions, vectors) nor gravity couples bilinearly to Q in the Cartan-Einstein theory, implying that no kinetic term for Q can be generated at the one-loop order by integrating out matter or gravitational fields. This is to be contrasted with torsion trace vector and skew symmetric torsion, whose kinetic terms are generated at one-loop order when scalar and fermionic matter is integrated out, respectively, thereby making them dynamical [18].

${Q^x}_{x(0)}$, ${Q^y}_{y(0)}$ and ${Q^x}_{y(0)} = {Q^y}_{x(0)}$. Inserting this into (4.22) and assuming the form for J^i as in (4.19) results in,

$$\Delta J^{x/y}_{(0)} = -\frac{2c}{\omega}\left[{Q^x}_{x/y(0)} J^{x/y}_{(0)} + {Q^x}_{y/x(0)} J^{y/x}_{(0)}\right], \Delta J^z_{(0)} = 0\,. \tag{4.24}$$

We thus see that the response of a passing Q wave resembles that of a gravitational wave, with two important differences: (1) a phase shift of $\pi/2$ characterizes the Q wave and (2) the response to a Q wave is inversely proportional to frequency.

4.4 The goldstone mode of scale symmetry breaking

In this section we want to analyse the theory we described in chapter 3, as it constitutes an example of a physical realisation of the scale symmetry breaking we conjectured several times in this thesis. The relevant question we want to answer is what sort of low energy signatures one can expect in this theories? It is well known, e.g. [106], that these appear in the form of Nambu-Goldstone bosons, which are typically massless excitations.

The description of the low energy theory for a given symmetry breaking pattern ($\mathcal{G} \to \mathcal{H}$), can be obtained through the construction of the Maurer-Cartan form for a representative element $\Omega \in \mathcal{G}/\mathcal{H}$ [107],

$$J_\mu \mathrm{d}x^\mu = \Omega^{-1}\mathrm{d}\Omega\,. \tag{4.25}$$

The case under consideration here is $SO(2,4) \to SO(1,3)$, which brings the full conformal group down to the Lorentz group. If at short distances nature behaves as a conformal field theory, defined at the critical point of some renormalization group flow, then we do expect the quantum state to be invariant under $SO(2,4)$, which is then broken as the theory flows away from the fixed point. The residual symmetry is that of quantum field theory vacuum, namely Lorentz invariance.

Denoting by $\Sigma^{mn}\,, K^m\,, D\,, P^m$ respectively the generators of Lorentz transformations, special conformal transformations, dilatations, and translations, one finds the following components of the Maurer-Cartan form [107],

$$[J_\mu]^{P^m} \equiv {}^P J^m_\mu = e^m_\mu\,, e^m_\mu e^n_\nu \eta_{mn} = g_{\mu\nu} \tag{4.26}$$

$$[J_\mu]^{\Sigma_{mn}} \equiv {}^\Sigma J^{mn}_\mu = S^{mn}_\mu - \left(e^m_\mu \mathcal{T}^n - e^n_\mu \mathcal{T}^m - 2e^m_\mu \chi^n + 2e^n_\mu \chi^m\right), \tag{4.27}$$

$$[J_\mu]^D \equiv {}^D J_\mu = \mathcal{T}_\mu - 2\chi_\mu\,, \tag{4.28}$$

$$[J_\mu]^{K_m} \equiv {}^K J^m_\mu = \nabla^L_\mu \chi^m + \mathcal{T}^m \chi_\mu + \left(\chi^2 - \mathcal{T}^\nu \chi_\nu\right) e^m_\mu + \left(\mathcal{T}_\mu - 2\chi_\mu\right)\chi^m\,, \tag{4.29}$$

where χ_m are the goldstone modes of special conformal transformations, $\mathcal{T}_\mu$ is the gauge field of local Weyl transformations, e^m_μ is the vierbein field, S^{mn}_μ

is its associated lorentz connection and $\nabla^L_\mu \chi^m = \partial_\mu \chi^m + S_\mu^{\ mn}\chi_n$ the Lorentz invariant derivative.

Note that, under the assumption that the spin connection S_μ^{mn} is torsion free, $\partial_{[\mu} e^m_{\nu]} + e_{m[\mu} S^{mn}_{\nu]} = 0$, we find the following torsion of the Lorentz component of the Maurer-Cartan form,

$$T^m_{\mu\nu} = \partial_{[\mu} e^m_{\nu]} + e_{n[\mu}{}^{\Sigma}J^{nm}_{\nu]} = \partial_{[\mu} e^m_{\nu]} + e_{n[\mu} J^{nm}_{\nu]} = e^m_{[\mu}\left(\mathcal{T}_{\nu]} - 2\chi_{\nu]}\right) \equiv e^m_{[\mu}\tilde{\mathcal{T}}_{\nu]}\,. \tag{4.30}$$

The procedure that is usually carried out to construct the low energy effective field theory, is to employ the so-called inverse Higgs constraint, which in our notation is equivalent to imposing $\tilde{\mathcal{T}}_\mu = 0$. This choice eliminates the torsion degrees of freedom, and is equivalent to integrating out the high energy, massive modes of the theory.

To elaborate on this, consider that the field $\mathcal{T}_\mu$ will become massive after symmetry breaking, with mass roughly given by the symmetry breaking scale. Integrating it out will thus yield corrections to the EFT that are suppressed by the symmetry breaking scale, therefore irrelevant at low energies. For this to hold, however, we must make sure that we work in an unitary gauge of sort, where the dilaton degree of freedom is sitting in some other field, typically in the metric field $g_{\mu\nu}$.

It is important to realise, at least in our case, that such a choice would require specifying a gauge, since a Weyl transformation will change $\tilde{\mathcal{T}}_\mu \to \tilde{\mathcal{T}}_\mu + \partial_\mu \log\Omega$. In gauge theories such as the one describing the electro-weak interaction, this is overcomed by defining the so-called unitary gauge, in which the goldstone modes decouple, and the gauge field only carries the massive vector polarization. However, the goldstone modes are still there, the gauge choice only means that they sit in the scalar field rather than in the vector field.

A similar notion exists in gravitational theories, as we argued in chapter 3, as experimental measurements are performed by assuming general relativity. This brings us to the notion of Einstein gauge, that is where the gravitational action takes the Einstein-Hilbert form.

Chosing the Einstein's gauge has one key advantage: in this gauge, since it is defined by requiring that the gravitational action is of the form,

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{M_P^2}{2}R\,,$$

the metric $g_{\mu\nu}$ propagating modes are traceless and transverse, which also means there does not exist a scalar propagating graviton [3]. Given that the combination of longitudinal torsion and scalar graviton that is gauge invariant, with respect to Weyl trasformations, is a linear combination of these two, setting the scalar graviton to zero (which must hold in the Einstein's gauge)

[3] Indeed, all scalar modes of the metric, such as the gravitational potentials $\Phi\,,\Psi$ are connected to their source via the gravitational constrains.

puts the physical goldstone mode in the longitudinal torsion, and one does not have to worry about the possibility of mixing [4].

This argument is only valid perturbatively, since there exist higher order interactions that might generate mixing, and also scalar matter field and condensates can interact with the metric trace mode and generate mixing. A more rigorous analysis should take into account thus the possibility of a mixing between those, by using the theory we develop here to generate the waveforms templates that are necessary for detection. Since the following analysis is a linear theory result, to quantitevely assest the possibility of a detection, we shall not worry about this complications, and leave it for future works on the subject.

4.5 Deriving the scalar torsion field equations

As an example of the theory describing the symmetry breaking pattern from last chapter, we will take the action (3.9) and use the gauge fixing (3.14) to obtain,

$$S = \int d^4x\sqrt{-g}\Bigg(\left(-\frac{\xi^2}{16\alpha}-\lambda\right)\left(\delta_{IJ}\phi^I\phi^J\right)^2+\frac{\xi}{8\alpha}M_P^2\delta_{IJ}\phi^I\phi^J-\frac{1}{16\alpha}M_P^4 \\ +\frac{M_P^2}{2}\left(\overset{\circ}{R}-6\cancel{\overset{\circ}{\nabla}{}^2\chi}+6\partial_\mu\chi\partial^\mu\chi\right)+\frac{1}{2}\delta_{IJ}\left(\partial_\mu+\partial_\mu\chi\right)\phi^I\left(\partial_\nu+\partial_\nu\chi\right)\phi^J\Bigg)\,, \tag{4.31}$$

where we dropped a total derivative $-6M_P^2\overset{\circ}{\nabla}{}^2\chi$. In studying this example, we note that here $\mathcal{T}_\mu = \partial_\mu\chi$, and χ is the goldstone mode that we are after.

Taking the trace of the Einstein's equations for this action, and using the equation of motion for the field χ, we must recover the gauge fixing condition (3.14). Indeed, by taking the equations of motion for χ, using the equations of motion for ϕ and the trace of the equations of motion for the metric, we arrive at,

$$\overset{\circ}{\Box}\chi = \frac{1}{6}\left(\overset{\circ}{R}+6\partial_\mu\chi\partial^\mu\chi-\frac{1}{4\alpha}M_P^2+\frac{\xi}{4\alpha}\phi^I\phi^J\delta_{IJ}\right)\,. \tag{4.32}$$

We could employ Einstein's equations once again, to obtain

$$\overset{\circ}{R}+6\partial_\mu\chi\partial^\mu\chi-\frac{1}{4\alpha}M_P^2 = \frac{T_\mu^{\mu,\phi}}{M_P^2}\,, \tag{4.33}$$

[4] Note that here we mean by Einstein's gauge, the choice of constant Planck mass. Should we study a theory with non-minimally coupled scalar fields as,

$$S = \int D^4x\sqrt{-g}\frac{\Phi^2}{2}R\,,$$

we would have a dynamical gravitational scalar.

which would lead to a equivalent form of (4.32),

$$\overset{\circ}{\Box}\chi = \frac{1}{6M_P^2}\left(T_\mu^{\mu,\phi} + \frac{\xi}{4\alpha}M_P^2\phi^I\phi^J\mathcal{G}_{IJ}\right), \tag{4.34}$$

where we set the geometrical quantity, $M_P^2\overset{\circ}{R} + 6M_P^2(\partial_\mu\chi)^2 - \frac{1}{4\alpha}M_P^4$ equal to its source in terms of the fields ϕ_I, in (4.34) represented as $T_\mu^{\mu,\phi}$.

In the theory under consideration, for $O(N)$ invariant scalars, Eq. (4.34) would read, ignoring possible anomalous terms coming from the Weyl anomaly,

$$\overset{\circ}{\Box}\chi = -\frac{1}{6M_P^2}\left(\nabla_\mu\phi_I\nabla^\mu\phi^I + 4\left(\lambda - \frac{\xi^2}{2\alpha}\right)\left(\mathcal{G}_{IJ}\phi^I\phi^J\right)^2 - \frac{\xi}{4\alpha}M_P^2\mathcal{G}_{IJ}\phi^I\phi^J\right), \tag{4.35}$$

which upon writing all the χ terms on the left, gives,

$$\begin{aligned}&\overset{\circ}{\Box}\chi + 2\frac{\phi_I}{6M_P^2}\partial_\mu\phi^I\partial^\mu\chi + \frac{\phi_I\phi^I}{6M_P^2}\partial_\mu\chi\partial^\mu\chi = \\ &= -\frac{1}{6M_P^2}\left(\partial_\mu\phi_I\partial^\mu\phi^I + 4\left(\lambda - \frac{\xi^2}{2\alpha}\right)\left(\mathcal{G}_{IJ}\phi^I\phi^J\right)^2 - \frac{\xi}{4\alpha}M_P^2\mathcal{G}_{IJ}\phi^I\phi^J\right).\end{aligned} \tag{4.36}$$

Eqs. (4.36) represent the field equation for the longitudinal mode of the torsion, which, as one can easily see, is massless and has a kinetic interaction with the scalar fields ϕ^I. This is a toy model and does not take into account possible anomalous terms that stem from the anomalous dimensions of operators in a interacting quantum theory (which are also a source for the mode χ). In what follows we attempt a more general formulation, which does not assume anything for the matter action, other than its classical scale invariance. Our working assumption shall remain that of spontaneous breaking of Weyl symmetry.

This should stem from a Weyl invariant action, $S[g_{\alpha\beta}, \mathcal{T}_\mu, \Psi]$, which in the broken symmetry phase can be evaluated in the Einstein's gauge, in which we obtain, upon setting $\mathcal{T}_\mu = \partial_\mu\chi$,

$$S_{\rm EF} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{\omega^2}{2}\left(\overset{\circ}{R} - 6\overset{\circ}{\Box}\chi + 6\partial_\mu\chi\partial^\mu\chi\right) + \mathcal{L}_m(\psi, \partial_\mu\chi, g_{\mu\nu}, \omega)\right]. \tag{4.37}$$

Varying this action with respect to ω, and then choosing the gauge $\omega \to M_P$ leads to the equation,

$$\left(\overset{\circ}{R} - 6\overset{\circ}{\Box}\chi + 6\partial_\mu\chi\partial^\mu\chi\right) = -\frac{1}{M_P}\frac{\partial\mathcal{L}_m(\psi, \partial_\mu\chi, g_{\mu\nu}, M_P)}{\partial M_P}, \tag{4.38}$$

which for the theory (4.31) is equivalent to the gauge fixing condition. Eq. (4.38) can be massaged upon using the Einstein's equations,

$$\overset{\circ}{R} + 6\partial_\mu\chi\partial^\mu\chi = \frac{2}{M_P^2}\frac{g^{\alpha\beta}}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\alpha\beta}}\,,$$

into the form,

$$\overset{\circ}{\Box}\chi = \frac{1}{6M_P^2}T_\mu^{\mu,m} + \frac{1}{6M_P}\frac{\partial\mathcal{L}_m(\psi\,,\partial_\mu\chi\,,g_{\mu\nu}\,,M_P)}{\partial M_P}\,. \tag{4.39}$$

Equation (4.39) together with the Einstein's equations,

$$\overset{\circ}{R}_{\mu\nu} - \frac{g_{\mu\nu}}{2}\overset{\circ}{R} + 6\partial_\mu\chi\partial_\nu\chi - 3g_{\mu\nu}\partial_\alpha\chi\partial^\alpha\chi = -\frac{2}{M_P^2}\frac{1}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \equiv -\frac{1}{M_P^2}T^m_{\mu\nu}\,, \tag{4.40}$$

constitues the full set of dynamical equations in the geometrical side of the theory.

In a semi-classical treatment, one can derive the right hand side of (4.39) from a quantum effective action, in which case, it will be built up of scalar condensates in the fashion of (4.36) and all other contributions must come from the anomaly, both gravitational or due to the renormalisation group flow of the coupling constants. Both these contributions are small in the great majority of physical situations, but might acquire large contributions in extreme conditions, such as in the vicinity of neutron stars or black holes.

Near black holes the space-time curvature can be strong enough for the terms $\propto R^2$ to contribute, while inside neutron stars we expect large QCD condensates to develop, which will source the dilaton field via the contribution,

$$\beta(\alpha_S)\mathrm{Tr}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}\,,$$

where $\beta(\alpha_S)$ is the beta function of the strong interaction coupling constant, α_S, and $\mathcal{F}_{\alpha\beta}$ is the $SU(3)$ field strength.

4.6 Propagation

In this section we discuss propagation of torsion trace, in the theory that we investigated in Chapter 3. We shall not discuss here skew symmetric torsion, since it is not detectable by standard gravitational wave instruments, nor the additional polarisation Q from q. (4.5), since no well motivated coupling of this object to matter exists in the literature.

The mode that we expect to propagate to very large distances is the longitudinal part of the torsion trace $\mathcal{T}_\mu$, since this mode is typically massless [25, 91]. Now upon setting $\mathcal{T}_\mu = \partial_\mu\theta$ we find,

$$\Box\theta = \frac{8\pi G_N}{c^4}\frac{T_\mu^\mu}{6}\,, \qquad \Box h_{ij} = \frac{8\pi G_N}{c^4}T_{ij}\,, \tag{4.41}$$

where for comparison we also wrote the equation for the transverse traceless tensor mode h_{ij}. We want to highlight some features of Eq. (4.41): first, we must require the underlying theory to be classically conformal, in order to comply with solar system tests of general relativity, see for example [96]. This would mean that the energy momentum trace on the right hand side of (4.41) is the anomalous trace, generated by quantum effects [25, 31] and thus will only be relevant if large curvature condensates, or possibly QCD condensates (*i.e.* that source an anomalous trace via the running of the coupling constants), are generated. This would put estimates of the amplitude of such a wave on the low side of the spectrum, still black holes and neutron stars can in principle support condensates large enough to produce a sizeable emission.

An analogous analysis to that of tensor waves leads to the solution for the scalar mode, which in the wave zone and to linear order reads,

$$\theta(t, \vec{r}\,) \simeq \frac{8\pi G_N}{12c^4} \frac{1}{\|\vec{r}\,\|} \frac{\mathrm{d}^2\mathcal{I}}{\mathrm{d}t^2}(ct - \|\vec{r}\,\|)\,, \tag{4.42}$$

where $\mathcal{I}_{ij}$ is the moment of inertia of the source, $\mathcal{I} = \mathcal{I}_{ij}\delta^{ij}$. Eq. (4.42) signals production and propagation of the scalar mode θ by the same sources as those of gravitational waves, namely the moment of inertia of the source. More precisely the source of this scalar disturbance in Eq. (4.42) is the trace of the moment of inertia, while the source of tensor modes is its traceless and transverse part. To get a scalar wave production it is not enough to have circular orbits, but it is required that the source system, other than quadrupole moment, has a monopole component. This implies that the scalar wave has amplitude proportional to the orbit eccentricity, and the ratio between scalar and tensor amplitudes is roughly, $\theta/h_{ij} \sim e^2/2$, where e is the source eccentricity. This means suppression by a factor of rougly 10, which renders it detectable by the next generation of gravitational waves observatories such as LISA and Einstein Telescope, albeit probably not by LIGO and VIRGO. Furthermore, due to the result of the last section, we can test whether torsion exists by looking at the *phase* of the displacement field. Indeed, given that the medium dispersion contributes negligibly, detecting a phase shift of $\pi/2$ between tensor and torsion waves would be a smoking gun for torsion. Finally, the longitudinal nature of this wave, namely $\Delta J_z \propto J_z$, provides an additional distinguishable feature from the usual gravitational wave signal, which will be detectable by gravitational waves observations with good angular resolution.

We should emphasize that the analysis presented in this section should be taken as an illustration on what kind of equation might govern propagation of torsion trace. On the other hand the analysis presented in section "Detection" is much more general and therefore it is not limited to the model presented in this section.

4.6.1 Detection at particle accelerators

Last chapter we saw that the scalar torsion trace, ϕ^0, according to Eq. (3.17). It will therefore also mix with the Higgs field, which would lead to an interacion potentially detectable at LHC. However, the strength of this coupling is gravitational, as one can understand from canonically normalizing the kinetic term for the scalar torsio,

$$\begin{aligned} &6M_P^2(\partial_\mu\phi^0)^2 \to (\partial_\mu\tilde{\phi}^0)^2\,,\\ &\implies \partial_\mu\phi^0 \mathrm{Tr} H^\dagger \partial^\mu H \to \frac{\partial_\mu\tilde{\phi}^0 \mathrm{Tr} H^\dagger \partial^\mu H}{\sqrt{6}M_P}\,, \end{aligned} \tag{4.43}$$

where Tr means taking the trace over the $SU(2)$ indices, which means that all the interactions mixing the Higgs field to scalar torsion are suppressed by $\frac{k}{M_P}$, where k is the momentum of the field (roughly the center of mass energy of a collision). This means that the cross section for the production of $\tilde{\phi}^0$ are heavily suppressed, and we reckon extremely hard to be resolved at particle accelerators.

Finally, the same reasoning implies that the transverse modes of torsion acquire a mass, of the order of the Planck mass, which will render them virtually undetectable by any means.

4.7 Summary and discussion

In this section we have shown that dynamical (propagating) torsion can be observed by conventional gravitational wave detectors. More precisely, torsion trace and torsion of mixed symmetry can be observed, while skew symmetric torsion cannot.

In sections 4.5–4.6 we discuss propagation of the torsion waves, and what polarisation we expect to be the most relevant, namely the longitudinal torsion trace. Indeed, this represents the goldstone mode of broken scale symmetry, and is therefore expected to be massless.

In section 4.3 we argued that this mode has a distinct longitudinal signal for the displacement vector $\Delta\vec{J} \propto \vec{J}$, a dependence on the source orbital eccentricity and a phase shift of $\pi/2$ with respect to the usual gravitational wave h_{ij}, all of which would render possible to distinguish the two signals. A successful detection still requires a precise characterisation of the signal, which is usually implemented by a collection of waveforms expected from typical sources. Although this analysis is beyond the scope of this thesis, we are confident it can be achieved by suitably adapting the usual post-Newtonian analysis.

To conclude, it is worth enlisting likely sources that could produce torsion waves of detectable strength. Since we assume torsion to be dynamical, the usual suspects - such as inflation, preheating, phase transitions and violent astrophysical events - can be invoked to be responsible for production.

Chapter 5

Quantum aspect of Weyl invariant interaction

5.1 Introduction

In this section we analyse some aspect of the scalar - vector - tensor theory defined for example by the action (3.6). Our aim is to better understand the role of the symmetry in a quantum theory, which translates to understanding the Ward identities that the symmetry requires. The case of Weyl symmetry is somewhat special, as the Ward identities that one can derive for a classically Weyl invariant action, are spontaneously broken by quantum effects. This fact is known in the literature as the conformal anomaly.

The discovery of conformal anomaly dates back to 1974 to the seminal work of Capper and Duff [23], in which the authors showed that the Ward identities of conformal symmetry are broken by the 1-loop quantum fluctuations. Next Capper and Duff found that the one-loop photon contributes anomalously to the graviton self-energy through the time-ordered energy-momentum tensor (TT) correlator on Minkowski space, $\langle T[T^{\mu}_{\mu}(x)T_{\sigma\lambda}(0)]\rangle \neq 0$. Their results show that, while the counter-terms proportional to $1/(D-4)$ respect Weyl symmetry, where D denotes the dimension of spacetime, the finite contribution does not, yielding an anomalous contribution to the TT correlator. In Ref. [17] a second type of anomaly, related to the Euler characteristic of the (Euclidean) manifold, appeared in the trace anomaly, given in four dimensions by the Gauss-Bonnet density, namely $\langle T^{\mu}_{\mu}\rangle \propto R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, where $R_{\mu\nu\lambda\sigma}$ is the curvature tensor and $R_{\mu\nu} = g^{\lambda\sigma}R_{\lambda\mu\sigma\nu}$, $R = g^{\mu\nu}R_{\mu\nu}$.

Such a spontaneous breaking of Weyl symmetry is often attributed to the anomalous (*i.e.* differrent than the classical scaling) running of coupling constants, induced by loop corrections to the effective action. We believe however, that there are two distinct kind of Weyl anomalies, the first being indeed induced by the running of the coupling constants, which is due to a spontaneous breaking of the global scale symmetry induced by radiative effects, and the second being type topological, as the Gauss-Bonnet anomaly, which is not induced by the running, and thus constitute a different beast.

Such distinction is important if one considers theories at the conformal fixed point of the renormalisation group flow. This is by defintion a point where the running ceases and quantum fluctuations become scale invariant.

Therefore, we expect the first kind of anomaly to disappear at the conformal fixed point. On the other hand the topological anomalies do not depend on the scale, and thus persist even at the conformal fixed point. They do not, however, induce running of the coupling constants.

We will in the first section of this chapter, that there exists a way of renormalising the theory explicitely showing that the topological terms are not anomalous, and that this is a consequence of introducing a compensating field for Weyl transformations. We use this technique to compute the one loop quantum effective action for the scalar-vector-tensor theory (3.6). We then compute the one loop corrected vertices [1],

$$\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}} = \frac{\delta^2 \Gamma_{eff}}{\delta \mathcal{T}_\alpha \delta \mathcal{T}_\beta}\,, \Gamma^{\mu\nu\beta}_{g\mathcal{T}} = \frac{\delta^2 \Gamma_{eff}}{\delta g_{\mu\nu} \delta \mathcal{T}_\beta}\,,$$

and study their properties in perturbative quantum field theory (namely, by taking the limit $g_{\mu\nu} \to \eta_{\mu\nu}\,, \mathcal{T}_\mu \to 0$).

Our results indicate that the anomalous corrections to the energy momentum trace are compensated by the quantum corrections to the dilatation current. Our computation evaluates the contributions to the energy momentum tensor and to the dilatation current, to linear order in the external fields $g_{\mu\nu}\,, \mathcal{T}_\mu$, showing that the anomalous contributions to the energy momentum and to the dilatation reciprocally cancel, at linear order. This shows that the cancellation is exact for the term appearing in the trace anomaly, proportional to $\Box R$, *i.e.* $\langle T^\mu_\mu \rangle \Box R$.

This is but the first step in understanding the Ward identities of this theory in the gravitational sector. This is because the anomaly contains contributions that go as $\propto R^2_{\alpha\beta\gamma\delta}$, which only appear, in the flat space limit $g_{\mu\nu} \to \eta_{\mu\nu}\,, \mathcal{T}_\mu \to 0$, at the level of 3-points functions. Although this computation is not included, the calculation of the 2-points functions suggests that the extended Ward identities (5.1) are in fact valid in any dimensions, and thus preserved by dimensional regularization. If this property is preserved at the level of the 3-point function, then the result would be much stronger, proving that no curvature squared term should appear in the anomaly for our extended theory.

5.1.1 Fundamental Ward identity

The idea we pursue in this thesis is to consider Weyl symmetry as a gauge transformation, which is compensated by a one-form (Weyl) field transforming as, $\mathcal{T}_\mu \to \mathcal{T}_\mu + \partial_\mu \log \Omega(x)$, where the Weyl transformation is defined by $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. As we show below, this then instigates the following modification of the fundamental Ward identity for Weyl symmetry,

$$\langle T^\mu_\mu \rangle + \langle \bar{\nabla}_\mu D^\mu \rangle = 0\,, \tag{5.1}$$

[1] To check our computations we used the Package-X addition to mathematica [108], which allows to compute 1-loop integrals in dimensional regularization.

where D^μ is the dilatation current that sources the Weyl field $\mathcal{T}_\mu$ and the brackets denote the time ordered product of operators. Equation (5.1) is the fundamental Ward identity for local Weyl symmetry. The identity simply states that there exists a current D^μ whose divergence equals to the trace of the energy-momentum tensor. A similar condition has been stated in [109] as a flat space-times requirement for a generic field theory to exhibit conformal symmetry.

For an interacting scalar field theory, for example the one defined in (3.6), the vacuum-to-vacuum scattering amplitude reads, [2]

$$\langle in|out\rangle = \int \mathcal{D}\phi\mathcal{D}\pi_\phi \exp\left\{ i\int \mathrm{d}^{D-1}\vec{x}\mathrm{d}t\big(\pi_\phi n^\mu\partial_\mu\phi - \mathcal{H}_\phi\big)\right\}\,, \tag{5.2}$$

where $n^\mu\partial_\mu\phi = \dot\phi$ if the spatial hypersurface Σ is chosen to be a constant time hypersurface, $\mathcal{H}_\phi(\phi,\pi_\phi) = \pi_\phi n^\mu\partial_\mu\phi - \sqrt{-g}\mathcal{L}_\phi(\phi,\partial_\mu\phi)$ is the Hamiltonian density and $\mathcal{L}_\phi$ is the Lagrangian density (which for now needs not be specified). With the definitions of scalar field and canonical momentum in mind we immediately see that the measure in (5.2) is Weyl invariant, since a field and its canonical momentum have opposite Weyl scaling. Thus the path integral in (5.2) and thus also the scattering amplitude must be Weyl invariant if $\mathcal{L}_\phi$ is.

The only thorny issue that might spoil conformal symmetry is related to the question of whether the path integral (5.2) is well defined. That indeed may pose a problem in the sense that the amplitude (5.2) is generally divergent and since any regularisation of (5.2) violates Weyl symmetry, it can make it 'anomalous.' However, as we will see in section 5.5, such anomalous terms are generated both in the energy momentum tensor and in the dilatation current, in such a way that they compensate each other.

It is worth remarking that in literature one often finds a path integral formulation in which the integration over the momentum is performed and in which Weyl symmetry of the path integral does not seem manifest. To show that this is not the case, let us perform the Gaussian integral over the (suitably shifted) momentum,

$$\begin{aligned}\int \mathcal{D}\tilde\pi_\phi \exp\left\{ i\int \mathrm{d}^D x\frac{\|n\|^2\tilde\pi_\phi^2}{\sqrt{-g}}\right\} &= \sqrt{\det\left(\sqrt{-g}\|n\|^{-2}\delta^D(x-y)\right)}\\ &= \prod_x\left(\frac{\sqrt{-g(x)}}{\|n(x)\|^2}\right)^{\frac12}.\end{aligned} \tag{5.3}$$

[2] For fermions the measure is,

$$\mathcal{D}\psi\mathcal{D}\pi_\psi = \mathcal{D}\psi\mathcal{D}\bar\psi\det\left(\sqrt{-g}\|n\|^{-2}n^\nu g_{\nu\mu}\gamma^\mu\right),$$

where the determinant is taken both on spinor indices and on spacetime continuous indices. The measure is both diffeomorphism and Weyl invariant.

With this result in mind, Eq. (5.2) can be written as,

$$\langle in|out\rangle = \int \bar{\mathcal{D}}\phi \, e^{iS_\phi} \,, \tag{5.4}$$

where $S_\phi = \int d^D x\sqrt{-g}\mathcal{L}_\phi$ and the barred measure is

$$\bar{\mathcal{D}}\phi = \prod_x d\phi(x) \left(\frac{\sqrt{-g(x)}}{\|n(x)\|^2} \right)^{\frac{1}{2}} \,, \tag{5.5}$$

which is obviously Weyl invariant. Note the dependence on the metric tensor in (5.5), which is usually omitted from the measure, but is essential for Weyl symmetry.

The vacuum-to-vacuum scattering amplitude is,

$$\langle in|out\rangle = \int \bar{\mathcal{D}}\phi \, e^{iS_\phi} \,, \tag{5.6}$$

where S_ϕ is a conformal scalar action, whose kinetic part is quadratic in the fields [3].

Requiring that infinitesimal Weyl transformations, $\Omega(x) \to 1 + \omega(x)$, under which the fields transform as,

$$\begin{gathered} \phi \to \phi' = \phi - \frac{D-2}{2}\omega\phi \,, \quad g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + 2\omega g_{\mu\nu} \,, \\ \Gamma^{\alpha}_{\ \mu\nu} \to \Gamma'^{\alpha}_{\ \mu\nu} = \Gamma^{\alpha}_{\ \mu\nu} + \delta^{\alpha}_{\ \mu}\partial_\nu\omega \,, \end{gathered} \tag{5.7}$$

do not change the *in-out* amplitude (5.6) yields,

$$\begin{aligned} \langle in|out\rangle &= \int \bar{\mathcal{D}}\phi' e^{iS'_\phi} \qquad (5.8) \\ &= \int \bar{\mathcal{D}}\phi e^{iS_\phi} \Bigg[1 + i\int \mathrm{d}^D x\sqrt{-g}\Bigg(-\frac{D-2}{2\sqrt{-g}}\frac{\delta S_\phi}{\delta\phi(x)}\omega(x)\phi(x) \\ &\quad + \frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}(x)}\omega(x) g^{\mu\nu} + \bar{\nabla}_\mu \left(\frac{1}{\sqrt{-g}}\frac{\delta S_\phi}{\delta \mathcal{T}_\mu(x)} \right)\omega(x) \Bigg) \Bigg] \,. \end{aligned}$$

Since this must be true for any arbitrary infinitesimal $\omega(x)$, Eq. (5.8) then implies,

$$\int \bar{\mathcal{D}}\phi e^{iS_\phi} \left(T^\mu_\mu + \bar{\nabla}_\mu D^\mu \right) = 0 \,, \tag{5.9}$$

[3] We need this requirement to be able to do the path integrals in (5.3), which become non gaussian for more generic theories. This is not necessarily an obstruction, but will modify the measure (5.5), which is why we do not include it here.

where,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{5.10}$$

$$D^{\mu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \mathcal{T}_{\mu}(x)}, \tag{5.11}$$

and we have used the Ehrenfest theorem, or the fact that, for any operator $\hat{\mathcal{O}}$ in the theory,

$$\left\langle \frac{\delta S_{\phi}}{\delta \phi(x)} \hat{\mathcal{O}} \right\rangle = 0 .$$

Upon dividing (5.9) by $\langle in|out \rangle$ we finally get,

$$\langle T^{\mu}_{\mu} \rangle + \langle \bar{\nabla}_{\mu} D^{\mu} \rangle = 0 , \tag{5.12}$$

proving thus (5.1). The angular brackets in (5.12) denote an expectation value of the time-ordered product and all the derivatives must be evaluated inside the time-ordered product.

If a theory is globally scale invariant, we would be led to the stronger requirement that $\langle T^{\mu}_{\mu} \rangle = 0$ and $\partial_{\mu} \langle D^{\mu} \rangle = 0$s (since for global scale transformations, $\partial_{\mu} \log \Omega = 0$), as pointed out again by Polchinski [109]. In such a case, at least for flat spacetimes, there exists a conserved current, the dilatation current, which is conserved, namely, $D^{\mu} = -T^{\mu}_{\nu} x^{\nu}$. From these observations it then follows that requiring $\langle T^{\mu}_{\mu} \rangle = 0$ is equivalent to demanding that the global scale transformation is a symmetry of the theory, which is not the case if the symmetry is *e.g.* broken by quantum effects. In other words, one can try to construct a classical action by using only the metric tensor and matter fields that is Weyl invariant. In constructing such a theory, however, one usually makes no distinction between global and local Weyl symmetry and a breaking of global scale symmetry implies a breaking of local conformal symmetry.

The crucial observation is that in flat space there *always* exists a dilatation current D^{μ} such that,

$$\partial_{\mu} D^{\mu} = -T^{\mu}_{\mu} , \tag{5.13}$$

which is divergence-free only if global scale symmetry is realised. Our proposal is to elevate the current D^{μ} to the source for the Weyl gauge field $\mathcal{T}_{\mu}$ on general curved spacetimes. If such a Weyl field exists it could be used to generate the source current *via*, $D^{\mu} = (-g)^{-1/2} \delta S / \delta \mathcal{T}_{\mu}$. Such a current is in general independent of the energy-momentum tensor and moreover – as we shall see – can be written as a local function of the fields. [4]

[4] The nonlocal expression, $D^{\mu}(x) = -(\partial^{\mu}/\Box) T^{\alpha}_{\alpha}(x)$, would obviously do. However, such forms for D^{μ} would be obtained by variation of the corresponding nonlocal effective actions. One could make these actions local by introducing an auxiliary field, whose physical meaning is that of a Weyl field $\mathcal{T}_{\mu}$. We may as well bypass the nonlocal step and from the very beginning work with a local formulation in which $\mathcal{T}_{\mu}$ exists as an independent field. That is the approach advocated here.

In [110, 111] and following works an energy momentum tensor satisfying,

$$\langle \Theta^{\mu}_{\mu} \rangle = \langle T^{\mu}_{\mu} + \nabla_{\mu} D^{\mu} \rangle = \sum_i \mu \frac{\partial \lambda_i}{\partial \mu} \frac{\partial \mathcal{L}_{eff}}{\partial \lambda_i} \,, \tag{5.14}$$

is constructed in the most general renormalisable field theory. In Eq. (5.14) λ_i are all the dimension full couplings in the Lagrangian, rescaled by the global renormalization scale μ, $T_{\mu\nu}$ the canonical energy momentum tensor, D^{μ} the dilatation current and $\mathcal{L}_{eff}$ the effective lagrangian. The identity (5.14) is shown to hold in [111] in any renormalised perturbation theory, to all orders in the coupling constants, that is even when the perturbative running of the coupling constants is included.

The expression for the full improved stress tensor reads, for the simplest scalar field theory,

$$\Theta_{\mu\nu} = T_{\mu\nu} - \frac{D-2}{4(D-1)} \left(\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box \right) \phi^2 \,, \tag{5.15}$$

from which we see it can be written in the notation used throughout this chapter as,

$$T_{\mu\nu} - \frac{1}{2(D-1)} \left(\nabla_{\mu} D_{\nu} - g_{\mu\nu} \nabla_{\lambda} D^{\lambda} \right) \,. \tag{5.16}$$

What this results imply, in the context of our discussion, is that, if no scale dependence is present in the coupling constants of the theory, it is always possible to find an energy momentum tensor, $T_{\mu\nu}$, and a current D^{μ}, which satisfy our fundamental Ward identity (5.1), at least in the flat space limit.

In [111] it is also shown that the stress tensor for the conformally coupled scalar (*i.e.* interaction in the form $\frac{\phi^2}{6} R$), having indeed the property of being traceless if the scalar field is massless, is the same we write in (5.15). In this particular theory, both the transversality, $\nabla_{\mu} \Theta^{\mu\nu}$ and the trace free condition becomes a consequence of the equations of motion. In a generic scalar tensor theory, however, although a tensor $\Theta_{\mu\nu}$ can be constructed having the property (5.14), it would not generically be an energy momentum tensor, because it would fail to be divergence free.

Elevating the dilatation current D^{μ} to the source for the Weyl gauge field $\mathcal{T}_{\mu}$, as we argue in this thesis, is the way to construct scale invariant theories in curved spaces, as it distinguishes the divergence free energy momentum tensor as the source of the metric field, from the dilatation current which sources the Weyl compensating field. This is relevant in understanding the role of scale symmetry in curved spaces and, as we shall see in the bulk of this section, it is a construction necessary to eliminate the unpleasant feature of the Weyl anomaly in space-times of non vanishing curvature. This might prove a necessary construction in particular in generalising the conformal field theories to curved space-times.

Although we expect theories possessing only dimensionless couplings to acquire scale dependence in renormalised perturbation theory, through the running of the coupling constants governed by the beta functions of the theory, there exist points in renormalisable field theories where this scale dependence disappears, the so-called conformal fixed points. Precisely at these points, the quantum theory should be describable in terms of a conformal field theory, and, as a consequence, the right hand side of (5.14) evaluates to zero. This is not the case on curved spaces, as the conformal anomaly acquires topological contributions, the most notable being the Euler density (5.62) (see section 5.3.1), which are independent on the beta functions of the theory. As we shall see in section 5.3.1 the introduction of the compensating Weyl field can render such terms innocuous, in the sense that they will not be anomalous and satisfy the identity (5.1), even though both T^{μ}_{μ}, and $\bar{\nabla}_{\mu}D^{\mu}$ acquire anomalous terms that do not exist in the classical theory (*i.e.* they are a consequence of divergent loop integrals).

Finally, in section 5.4 we will discuss how the longitudinal component of torsion can be used in renormalised perturbation theory to render the *off-shell* effective action conformally invariant, and how this procedure naturally leads to the curved space-times generalisation of the identity (5.14). As demonstrated in [111] this is a consequence of the equations of motion of the fields.

There are several operators for which D^{μ} is non trivial, for example all dimension four curvature operators with torsion, the scalar field kinetic terms and the non minimal couplings, as in (3.6). All these contributions can get sourced by a non vanishing energy-momentum tensor trace, such to respect the identity (5.1).

In order to see that the dilatation current naturally arises and that it can be written as a local function of the fields, let us consider an interacting, scale-invariant field theory (in $D = 4$) of N scalar fields,

$$S_{\{\phi^a\},N} = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}\zeta_{ab}\partial_{\mu}\phi^a\partial^{\mu}\phi^b + \frac{\lambda_{abcd}}{4}\phi^a\phi^b\phi^c\phi^d + \frac{\xi_{ab}}{2}\phi^a\phi^b R\right), \tag{5.17}$$

where ζ_{ab}, ξ_{ab} and λ_{abcd} are constants. It is easy to show that the trace of the energy-momentum tensor, $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$, satisfies [5],

$$T^{\mu}_{\mu} = \nabla_{\mu}\left[\left(\zeta_{ab} + 12\xi_{ab}\right)\phi^a\partial^{\mu}\phi^b\right] \implies D^{\mu} = -\left(\zeta_{ab} + 12\xi_{ab}\right)\phi^a\partial^{\mu}\phi^b\,. \tag{5.18}$$

Hence our prescription for the dilatation source, namely
$D^{\mu} = (-g)^{-1/2}\delta S/\delta\mathcal{T}_{\mu}$, yields naturally to the modification, $\partial_{\mu} \to \bar{\nabla}_{\mu}$, $R \to \bar{R}$, in the action (5.17), where $\bar{\nabla}_{\mu}$ is the conformal covariant derivative and $\bar{R}$ is the curvature scalar with torsion trace. We are then led to the action (3.6) generalised to N interacting scalars with non-minimal coupling to the curvature scalar and quartic interactions. Then for all values of ζ_{ab} and ξ_{ab} $(a, b, = 1, \ldots, N)$ we would have a Weyl invariant action whose energy momentum tensor is the divergence of a vector current.

[5] Note how, for the conformally coupled scalars, namely for $\zeta_{ab} = -12\xi_{ab}$, the dilatation current vanishes.

5.2 Riemann Normal coordinates

In this section we derive the metric expansion in Riemann Normal Coordinate in the case in which torsion consists purely of its trace, *i.e.* $T^{\lambda}{}_{\mu\nu} = \delta^{\lambda}_{[\mu}\mathcal{T}_{\nu]}$. The definition of Riemann Normal Coordinates is that geodesics in this coordinate system are locally straight lines, that is that the geodesic equation at the origin of Riemann Normal Coordinates reads,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}\Big|_{x=x_0} = 0 \implies \Gamma^{\lambda}{}_{(\mu\nu)}\Big|_{x=x_0} = 0\,. \tag{5.19}$$

However, we can immediately verify that this is not a consistent choice, from the point of view of conformal invariance, as changing conformal frame will change the geodesic tangent vector as,

$$\mathrm{d}x^\mu/\mathrm{d}\tau \to \Omega^{-1}(x)\mathrm{d}x^\mu/\mathrm{d}\tau\,.$$

Hence, one can only impose Eq.(5.19) in one frame, but not consistently in all conformal frames.

To solve this problem, let us define on the curve $x^\mu(\tau) = \gamma(\tau)\,, \mathrm{d}x^\mu/\mathrm{d}\tau = \dot{\gamma}^\mu$,

$$\chi(\tau,\tau_0;\gamma) = \chi(x,x_0;\gamma) = \exp\Big(\int_{x_0;\gamma}^{x} \mathrm{d}\tau\, \mathcal{T}_\mu \mathrm{d}x^\mu/\mathrm{d}\tau\Big)\,, \tag{5.20}$$

where the integral $\int_{x_0;\gamma}^{x}\mathrm{d}\tau$ is evaluated on the geodesic $\gamma(\tau)$, and τ_μ is the torsion trace. We can define, given two points x and x_0, the bi-scalar $\chi(x,x_0;\gamma)$ as in Eq.(5.20) where $\gamma(\tau)$ is a geodesic. Locally, the bi-scalar $\chi(x,x_0;\gamma)$ is well defined, as there will be only one geodesic linking x and x_0.

We notice, however, that in the expression (5.20) might become ill defined at large distances: given two curves $\gamma_1\,, \gamma_2$, and denoting by $\Sigma[\gamma_1,\gamma_2]$ a surface whose border is $\gamma_1 \cup \gamma_2$, it is clear that,

$$\frac{\chi(x,x_0;\gamma_1)}{\chi(x,x_0;\gamma_2)} = \exp\left(\int_{\Sigma[\gamma_1,\gamma_2]} \mathrm{d}x^\mu \mathrm{d}x^\nu \mathcal{T}_{\mu\nu}\right)\,, \mathcal{T}_{\mu\nu} = \partial_\mu \mathcal{T}_\nu - \partial_\nu \mathcal{T}_\mu\,, \tag{5.21}$$

which shows that one can pick up a phase, proportional to the flux of $\mathcal{T}_{\mu\nu}$, in the definition of the bi-scalar (5.20). This is not so important for the purpose of UV regularization, as on short enough scales we can choose to use the unique geodesic that links $x\,, x_0$. It is clear that the bi-scalar $\chi(x,x_0)$ becomes globally well defined (up to a multiplicative constant) if $\mathcal{T}_{\mu\nu} = 0$, *i.e.* if $\mathcal{T}_\mu$ is purely longitudinal.

The bi-scalar $\chi(x,x_0)$ gives us the notion of a geometrical scalar field, as performing conformal transformation changes $T_\mu \to T_\mu + \partial_\mu \log\Omega$, such that

$$\chi(x,x_0;\gamma) \to \Omega(x)\chi(x,x_0;\gamma)\Omega^{-1}(x_0)\,, \tag{5.22}$$

for any curve γ on which it is defined. Finally note that $\lim\limits_{x\to x_0} \chi(x,x_0) = 1$ in all frames and that, given the definition of $\bar{\nabla}_\mu$ from Eq. (2.46) we have

$$\dot{\gamma}^\mu \bar{\nabla}^x_\mu \chi(x,x_0;\gamma)\Big|_{x\to x_0} = -\dot{\gamma}^\mu \bar{\nabla}^{x_0}_\mu \chi(x,x_0;\gamma)\Big|_{x\to x_0} = 0\,,$$

where $\dot{\gamma}$ is the tangent vector to the curve γ.

With this machinery, let us define a conformal geodesic tangent vector as,

$$\dot{\Gamma}^\mu = \chi(\tau,\tau_0;\gamma)\dot{\gamma}^\mu = \chi(\tau,\tau_0)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\,, \tag{5.23}$$

note that $\dot{\Gamma}^\mu \to \Omega^{-1}(\tau_0)\dot{\Gamma}^\mu = \Omega^{-1}(x_0)\dot{\Gamma}^\mu$ under conformal rescaling. Therefore it makes sense to demand that $\frac{\mathrm{d}\dot{\Gamma}^\mu}{\mathrm{d}\tau}\big|_{x=x_0} = 0$, as this will be true in any conformal frame.

Next, using this definitions, we reparametrise the geodesic equation, defining a notion of invariant proper time as,

$$\mathrm{d}\bar{\tau} = \chi^{-1}(x,x_0)\mathrm{d}\tau \implies \dot{\Gamma}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\bar{\tau}}\,, \tag{5.24}$$

it is a simple matter of algebra to rewrite the geodesic equation as,

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\bar{\tau}^2} + \Gamma^\mu{}_{(\alpha\beta)}\dot{\Gamma}^\alpha\dot{\Gamma}^\beta - \chi(x,x_0)\frac{\mathrm{d}\chi}{\mathrm{d}\tau}\dot{\Gamma}^\mu = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\bar{\tau}^2} + \Gamma^\mu{}_{(\alpha\beta)}\dot{\Gamma}^\alpha\dot{\Gamma}^\beta - T_\alpha\dot{\Gamma}^\alpha\dot{\Gamma}^\mu\,, \tag{5.25}$$

where we used that,

$$\chi(x,x_0)\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = \chi(x,x_0)T_\alpha\dot{\gamma}^\alpha = T_\alpha\dot{\Gamma}^\alpha\,.$$

Finally, note that Eq.(5.25) can be written as,

$$\bar{\nabla}_{\dot{\Gamma}}\dot{\Gamma} = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\bar{\tau}^2} + \left(\Gamma^\mu{}_{(\alpha\beta)} - \delta^\mu_{(\alpha}T_{\beta)}\right)\dot{\Gamma}^\alpha\dot{\Gamma}^\beta = 0\,, \tag{5.26}$$

where $\bar{\nabla}$ is the conformal covariant derivative defined in (2.46), upon noticing that, for $\dot{\Gamma}$, $w_g - w = -1$.

We now demand that the geodesic equation rewritten in conformal time, Eq.(5.26), has straight line solutions in Riemann Normal Coordinates at the point x_0. Thus, we have that at x_0,

$$\left(\Gamma^\lambda{}_{(\mu\nu)} - \delta^\lambda_{(\mu}T_{\nu)}\right)\big|_{x=x_0} = 0\,. \tag{5.27}$$

Note that (5.27) can be always be achieved by a coordinate transformation, due to the non linear transformation law for the Christoffel connection.

Let us stress that the redefinition Eq. (5.24) is purely a reparametrisation of proper time, and as such it does not change the Jacobi Fields orthogonal to the geodesics. As a consequence, the Jacobi equation can be rewritten as,

$$\bar{\nabla}_{\dot{\Gamma}}\bar{\nabla}_{\dot{\Gamma}}J = \bar{R}(\dot{\Gamma}, J)\dot{\Gamma}\,. \tag{5.28}$$

Note that (5.28) coincides with Eq. (2.27), but we incorporated the terms depending on the torsion trace $\mathcal{T}_\mu$ in the definition of the conformal derivative, $\bar{\nabla}$ rather than writing them explicitely.

This is a powerful technique, since the rest of the derivation of Riemann normal coordinates can be performed using solely the commutation relations of $\bar{\nabla}$. Since the geodesics are locally straight lines, it means that in RNC coordinate system,

$$y^\mu(\bar{\tau}) = x^\mu - x_0{}^\mu = \dot{\Gamma}^\mu\bar{\tau} = \alpha^\mu\bar{\tau}\,, \tag{5.29}$$

where α^μ is a constant vector. This means that the Jacobi fields can be written as $j^\mu = \bar{\tau}\beta^\mu$, where β^μ is a constant.

We now Taylor expand the function on the geodesic curve γ,

$$f(\bar{\tau}) = \chi^{-2}(x(\bar{\tau}), x_0; \gamma)g(J, J)\,, \tag{5.30}$$

noting that (5.30) is Weyl invariant up to a constant w.r.t. $\bar{\tau}$, we have that:

$$\begin{aligned} f(\bar{\tau}) &= \sum_{n=0}^{\infty} \frac{\mathrm{d}^{(n)}}{\mathrm{d}\bar{\tau}^{(n)}} f\bigg|_{x=x_0} \frac{\bar{\tau}^n}{n!} = \sum_{n=0}^{\infty} \bar{\nabla}^n_{\dot{\Gamma}} f\bigg|_{x=x_0} \frac{\bar{\tau}^n}{n!} \\ &= \sum_{n=0}^{\infty} \bar{\nabla}_{\mu_1}\cdots\bar{\nabla}_{\mu_n} f\bigg|_{x=x_0} \dot{\Gamma}^{\mu_1}\cdots\dot{\Gamma}^{\mu_n}\frac{\bar{\tau}^n}{n!}\,. \end{aligned} \tag{5.31}$$

where we made substantial use of the geodesic equation (5.26). And, since by costruction, $\bar{\nabla}_{\dot{\Gamma}}\chi(x(\bar{\tau}), x_0) = 0$, and $\chi(x_0, x_0) = 1$ we can rewrite (5.31) as,

$$\begin{aligned} f(\bar{\tau}) = \chi^{-2}(x(\bar{\tau}), x_0)g(J, J) &= \sum_{n=0}^{\infty} \bar{\nabla}^n_{\dot{\Gamma}} g(J, J)\bigg|_{x=x_0} \frac{\bar{\tau}^n}{n!} \\ \Longleftrightarrow g(J, J) = \chi^2(x(\bar{\tau}), x_0) &\sum_{n=0}^{\infty} \bar{\nabla}^n_{\dot{\Gamma}} g(J, J)\bigg|_{x=x_0} \frac{\bar{\tau}^n}{n!}\,, \end{aligned} \tag{5.32}$$

Now, since the solution of (5.28) in this coordinate system is, $J = \bar{\tau}\beta$, we have that

$$\bar{\nabla}_{\dot{\Gamma}}J \overset{\bar{\tau}\to 0}{=} \beta\,, J \overset{\bar{\tau}\to 0}{=} 0\,, \tag{5.33}$$

where $x(\bar{\tau} = 0) = x_0$. Note also that the Jacobi equation gives the second conformal derivative along the geodesic of J. Thus, we can expand the metric in a manifestly Weyl and coordinate invariant form. This is done more specifically by explicitely computing the coefficients of the Taylor series (5.32) using the Jacobi equation (5.28), and its derivatives, evaluated at x_0. For example, one can immediately show that,

$$\bar{\nabla}_{\dot{\Gamma}}\bar{\nabla}_{\dot{\Gamma}}J\bigg|_{x=x_0} = 0\,,\ \bar{\nabla}_{\dot{\Gamma}}\bar{\nabla}_{\dot{\Gamma}}\bar{\nabla}_{\dot{\Gamma}}J\bigg|_{x=x_0} = \bar{R}(\alpha, \beta)\alpha\,, \cdots\,.$$

Finally, one can collect terms in the Taylor series (5.32) that multiply $\beta^\mu\beta^\nu$, and read of the expansion from the right hand side, grouping terms using the relation (5.29).

After some calculation we find that the metric expansion up to order 4 is,

$$g_{\mu\nu} = \chi^2(x, x_0; \gamma) \times C(x_0) \Bigg[\eta_{\mu\nu} - \frac{1}{3}\bar{R}_{\alpha(\mu\nu)\beta}\, y^\alpha y^\beta - \frac{1}{6}\bar{\nabla}_\alpha \bar{R}_{\beta(\mu\nu)\gamma}\, y^\alpha y^\beta y^\gamma + \left(-\frac{1}{20}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{R}_{\gamma(\mu\nu)\delta} + \frac{1}{60}\bar{R}_{\alpha(\mu|\beta\lambda}\bar{R}^\lambda{}_{\gamma|\nu)\delta} + \frac{1}{36}\bar{R}_{\lambda\alpha\mu\beta}\bar{R}^\lambda{}_{\gamma\nu\delta} \right) y^\alpha y^\beta y^\gamma y^\delta \Bigg] . \tag{5.34}$$

Note that each term in this expansion, which we can write as

$$A^{(n)}_{\mu\nu:\alpha_1\cdots\alpha_n} y^{\alpha_1} \cdots y^{\alpha_n}$$

satisfies, the following identity, which we will need later,

$$(i) \;\; A^{(n)}_{\mu(\nu:\alpha_1\cdots\alpha_n)} = 0 , \tag{5.35}$$

since it contains at least one pair of antisymmetric indices, as a consequence of the antisymmetry of the curvature tensor, $\bar{R}_{\alpha\mu\nu\beta} = -\bar{R}_{\mu\alpha\nu\beta} = \bar{R}_{\mu\alpha\beta\nu}$. This concludes the derivation of the Weyl invariant expansion of the metric.

5.3 UV expansion of scalar green's functions

We now want to use the expansion (5.34) to construct a short distance expansion for the propagator of the scalar field in the theory (3.6).For this purpose, consider a scalar field conformally coupled, it satisfies the one point function equation of motion,

$$\left[\bar{\nabla}_\mu\bar{\nabla}^\mu + \xi\bar{R}\right]\phi = 0 , \tag{5.36}$$

from which we infer the propagator equation,

$$\left[\bar{\nabla}_\mu\bar{\nabla}^\mu + \xi\bar{R}\right] G_F(x; x') = \frac{\delta(x - x')}{\sqrt{-g(x)}} . \tag{5.37}$$

Because of Eq. (5.34) we can write the metric as

$$g_{\mu\nu} = \chi(x, x'; \gamma)\hat{g}_{\mu\nu} , \; \sqrt{-g}g^{\mu\nu} = \chi^{\frac{D-2}{2}}\sqrt{-\hat{g}}\hat{g}^{\mu\nu} , \tag{5.38}$$

where $\hat{g}_{\mu\nu}$ is conformally invariant up to a constant w.r.t. x, that is $\hat{g}_{\mu\nu} \to f(x')\hat{g}_{\mu\nu}$ under conformal transformations. Upon noticing this fact, we can do the same for $G_F(x; x')$, and define the conformally invariant-at-x propagator,

$$\hat{G}_F(x; x') = \chi(x; x')^{(D-2)/2} G_F(x; x') , \tag{5.39}$$

which scales as a field of energy dimension $D-2$ at x', but does not scale at x. If we insert Eq. (5.38) and Eq. (5.39) into equation (5.37), we can rewrite it as,

$$\partial_\mu\left(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu\hat{G}_F(x;x')\right)+\left[\frac{\sqrt{-\hat{g}}}{\chi(x;x')^{-(D+2)/2}}\bar{\nabla}_\mu\bar{\nabla}^\mu\chi(x;x')^{-(D-2)/2}\right.$$
$$\left.+\xi\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\bar{R}_{\mu\nu}\right]\hat{G}_F(x;x')=\delta^D(x-x')\,. \tag{5.40}$$

Note Eq. (5.40) is an exact equation, since $\chi(x;x')\delta^D(x-x')=\delta^D(x-x')$, and also that Eq. (5.40) is conformally invariant both at x and at x'. To show this explicitly, we must recall that $\hat{g}_{\mu\nu}=\chi(x;x')^{-2}g_{\mu\nu}$, and as such scales at x' but not at x. The same applies to $\hat{G}_F$.

Now, we can simplify this equation further by defining

$$\hat{G}_F(x;x')=(-\hat{g})^{-\frac{1}{4}}\hat{\mathcal{G}}_F(x;x')\,,$$

we can rewrite the propagator equation (5.40) as,

$$(-\hat{g})^{\frac{1}{4}}\partial_\mu\left(\hat{g}^{\mu\nu}\partial_\nu\hat{\mathcal{G}}_F(x;x')\right)+\left[\frac{(-\hat{g})^{\frac{1}{4}}}{\chi(x;x')^{-(D+2)/2}}\bar{\nabla}_\mu\bar{\nabla}^\mu\chi(x;x')^{-(D-2)/2}\right.$$
$$\left.+\xi(-\hat{g})^{\frac{1}{4}}\hat{g}^{\mu\nu}\bar{R}_{\mu\nu}+\partial_\mu\left(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu(-\hat{g})^{-\frac{1}{4}}\right)\right]\hat{\mathcal{G}}_F(x;x')=\delta^D(x-x')\,. \tag{5.41}$$

Consider now only the first term in Eq. (5.41), expanded in Riemann normal coordinate about x' (with $y=x-x'$) would read:

$$\partial_\mu\left(\hat{g}^{\mu\nu}\partial_\nu\hat{\mathcal{G}}_F(x;x')\right)=\partial_\mu\left(\sum_{n=0}^{\infty}A^{(n):\mu\nu}_{\alpha_1\cdots\alpha_n}y^{\alpha_1}\cdots y^{\alpha_n}\partial_\nu\hat{\mathcal{G}}_F(x;x')\right), \tag{5.42}$$

if we now postulate that $\hat{\mathcal{G}}_F(x;x')=\hat{\mathcal{G}}_F(y^2)$, that is the state of the field respects Lorentz invariance, which is true in particular for the vacuum state, this would simplify as

$$\partial_\mu\left(\hat{g}^{\mu\nu}\partial_\nu\hat{\mathcal{G}}_F(x;x')\right)=\eta^{\mu\nu}\partial_\mu\partial_\nu\hat{\mathcal{G}}_F(y^2)\,, \tag{5.43}$$

where we used the identity in (5.35). Using Eq.(5.43), we can finally write the propagator equation (5.37) in Riemann Normal Coordinate, as

$$\partial_\mu\left(\eta^{\mu\nu}\partial_\nu\hat{\mathcal{G}}_F(x;x')\right)+\left[\frac{1}{\chi(x;x')^{-(D+2)/2}}\bar{\nabla}_\mu\bar{\nabla}^\mu\chi(x;x')^{-(D-2)/2}\right.$$
$$\left.+\xi\hat{g}^{\mu\nu}\bar{R}_{\mu\nu}+\frac{1}{(-\hat{g})^{\frac{1}{4}}}\partial_\mu\left(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu(-\hat{g})^{-\frac{1}{4}}\right)\right]\hat{\mathcal{G}}_F(x;x')=\frac{\delta^D(x-x')}{\hat{g}^{\frac{1}{4}}}\,. \tag{5.44}$$

Note that the simplifications that lead to Eq. (5.44) are a consequence of Lorentz invariance of the vacuum state, and that $G_F(x,x') = G_F(\|x-x'\|)$. This is not stricly speaking true, since $G_F(x,x')$ can contain dependence on x', but this dependence is assumed to be small, since we want to expand $G_F(x,x')$ in the limit of large difference momenta, or $\partial_y \gg \partial_{x'}$.

Eq. (5.44) is therefore exact up to corrections $\frac{\|\bar{\nabla}_{x'}G\|}{\|\bar{\nabla}_x G\|}$, however, its complete solution can be only written in terms of non local operators. Since our goal is to identify the divergent terms in (5.44), and renormalize the effective action by introducing local counter terms, we want to expand (5.44) in terms of local operators-at-x' and their derivatives.

To do this, we use the following statement, which is the same property we used to prove Eq. (5.31), concretely that: a conformally invariant-at-x function can be expanded in terms of its conformally invariant derivatives in the coordinates defined in this chapter. In formulas,

$$f(x) = \sum_{n=0}^{\infty} \bar{\nabla}_{\mu_1} \cdots \bar{\nabla}_{\mu_n} f \Big|_{x=x_0} \frac{y^{\mu_1} \cdots y^{\mu_n}}{n!} . \tag{5.45}$$

One can also prove this directly, by using the vanishing of the generalised connection condition (5.27) at point x_0. Eq. (5.45) can be applied to the functiona $\hat{g}^{\mu\nu}\bar{R}_{\mu\nu}$, and shows how it can be expanded in Riemann Normal Coordinates, which will make use of Eq. (5.34).

Still missing is an expansion for $\chi(x, x_0; \gamma)$, which one can find by simply expanding $\chi(x(\bar{\tau}), x_0; \gamma)$ as a power series in τ. This would lead to the expansions, up to operator of order four,

$$\bar{\nabla}_\mu \chi \bar{\nabla}^\mu \chi = -\frac{1}{4} \mathcal{T}_{\alpha\mu} \mathcal{T}^{\mu}{}_{\beta} y^\alpha y^\beta , \tag{5.46}$$

$$\begin{aligned} \chi \bar{\nabla}_\mu \bar{\nabla}^\mu \chi &= \frac{1}{3} \bar{\nabla}_\mu \mathcal{T}^{\mu}{}_{\nu} y^\nu + \frac{1}{8} \left(\bar{\nabla}_\mu \bar{\nabla}_\alpha \mathcal{T}^{\mu}{}_{\beta} + \bar{\nabla}_\alpha \bar{\nabla}_\mu \mathcal{T}^{\mu}{}_{\beta} \right) y^\alpha y^\beta \\ &\quad - \frac{1}{4} \mathcal{T}_{\alpha\mu} \mathcal{T}^{\mu}{}_{\beta} y^\alpha y^\beta , \end{aligned} \tag{5.47}$$

where as usual $\mathcal{F}_{\mu\nu} = 2\partial_{[\mu} \mathcal{T}_{\nu]}$.

Now we have all the ingredients to write Eq. (5.44) as a power series in y^n, we would find, keeping at most geometrical operators of dimension four,

$$\begin{aligned}
&\left[\eta^{\mu\nu}\partial_\mu\partial_\nu + \left(\xi - \frac{1}{6}\right)\bar{R}\right]\hat{\mathcal{G}}_F \\
&+ \left[\left(\xi - \frac{1}{6}\right)\bar{\nabla}_\alpha\bar{R} - \frac{D-2}{6}\bar{\nabla}_\mu F^\mu{}_\alpha\right] y^\alpha\hat{\mathcal{G}}_F + \mathcal{A}_{\alpha\beta}y^\alpha y^\beta\hat{\mathcal{G}}_F = \frac{\delta^D(y)}{\hat{g}^{\frac{1}{4}}}\,, \\
&\mathcal{A}_{\alpha\beta} = \frac{1}{2}\left(\xi - \frac{1}{6}\right)\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{R} + \frac{1}{120}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{R} - \frac{1}{30}\bar{R}_{(\alpha\lambda)}\bar{R}^{(\lambda}{}_{\beta)} + \frac{1}{40}\bar{\nabla}_\lambda\bar{\nabla}^\lambda\bar{R}_{\alpha\beta} \\
&- \frac{1}{20}\bar{R}_{[\alpha\lambda]}\bar{R}^\lambda{}_\beta + \frac{1}{60}\bar{R}_{(\kappa\lambda)}\bar{R}^\kappa{}_\alpha{}^\lambda{}_\beta \\
&+ \frac{1}{240}\left(\bar{R}_{\sigma\mu\alpha\lambda}\bar{R}^{\sigma\mu}{}_\beta{}^\lambda + \bar{R}_{\sigma\alpha\mu\lambda}\bar{R}^\sigma{}_\beta{}^{\mu\lambda} + 2\bar{R}_{\sigma\alpha\mu\lambda}\bar{R}^\sigma{}_\beta{}^{\mu\lambda}\right) \\
&- \left(\frac{D-2}{4}\right)^2 F_{\alpha\mu}F^\mu{}_\beta - \frac{D-2}{16}\left(\bar{\nabla}_\mu\bar{\nabla}_\alpha F^\mu{}_\beta + \bar{\nabla}_\alpha\bar{\nabla}_\mu F^\mu{}_\beta\right)\,,
\end{aligned} \tag{5.48}$$

where everything is evaluated at x', and the derivative are with respect to y.

Note that Eq. (5.48) can be written in fourier space, where the fourier transform is defined with respect to the coordinate $x^\mu - x'^\mu$,

$$\begin{aligned}
&\left[\eta^{\mu\nu}k_\mu k_\nu + M(x') + iM^{(1)}_\alpha(x')\partial^\alpha - \mathcal{A}_{\alpha\beta}(x')\partial^\alpha\partial^\beta\right]\hat{\mathcal{G}}_F(k) = 1\,, \\
&\text{with } M(x') = \left(\xi - \frac{1}{6}\right)\bar{R}(x')
\end{aligned} \tag{5.49}$$

where now $\partial^\alpha = \frac{\partial}{\partial k_\alpha}$. Since $M(x')$ is constant with respect to k, we can write the perturbative solution,

$$\begin{aligned}
\hat{\mathcal{G}}_F(k) = &\frac{1}{k^2 + M(x')} - \frac{iM^{(1)}_\alpha(x')}{k^2 + M(x')}\partial^\alpha\frac{1}{k^2 + M(x')} + \\
&+ \frac{\mathcal{A}_{\alpha\beta}(x')}{k^2 + M(x')}\partial^\alpha\partial^\beta\frac{1}{k^2 + M(x')} + \mathcal{O}\left(\frac{1}{k^8}\right)\,.
\end{aligned} \tag{5.50}$$

Since we are constructig an UV expansion for the scalar field propagator, we should ask ourselves what sort of expansion are we devising. We want to infer the behaviour of the field ϕ at short distances or large momentum. Usually, this can be done by requiring that $k \gg m$, where m is the mass of the field. However, since we are considering the case of conformally coupled scalars, we do not have at our disposal any intrinsic comparing scale. However, we can always compare the momentum of the field, that is k, with the momentum of the space-time, that is $R(x')$. This assumes that R can be expanded around a constant term, such that $R(x) \simeq R(x') + \partial_\mu R(x')y^\mu$, with $R(x') \neq 0$. $R(x')$ is then the scale we are using in our expansion to define what large momentum means.

The solution (5.50) is constructed on this principle, and can therefore be used to reliably calculate UV divergences. The fact that $M(x')$ appears on the denominator is just the result of a re-summation. In fact, we could have as well expanded, rather than in $\frac{1}{k^2+M(x')}$, in $\frac{1}{k^2}$. However, we would have found that this generates infinitely more terms in the series analougos to (5.50), re-summing all those contributions would lead to the expression (5.50). We choose to re-sum these contributions to render the integrals in k we are going to perform shortly finite in the IR.

By using the identities,

$$\begin{aligned}
\frac{1}{k^2+M(x')}\partial^\alpha\frac{1}{k^2+M(x')} &= \frac{1}{2}\partial^\alpha\frac{1}{(k^2+M(x'))^2}, \\
\frac{1}{k^2+M(x')}\partial^\alpha\partial^\beta\frac{1}{k^2+M(x')} &= \frac{1}{3}\partial^\alpha\partial^\beta\frac{1}{(k^2+M(x'))^2} - \frac{2}{3}\frac{\eta^{\alpha\beta}}{(k^2+M(x'))^3}.
\end{aligned} \tag{5.51}$$

We can rewrite (5.50) as,

$$\begin{aligned}
\hat{\mathcal{G}}_F(k) = {}& \frac{1}{k^2+M(x')} - \frac{i}{2}M^{(1)}_\alpha(x')\,\partial^\alpha\frac{1}{(k^2+M(x'))^2} + \\
&+ \frac{1}{3}\mathcal{A}_{\alpha\beta}(x')\partial^\alpha\partial^\beta\frac{1}{(k^2+M(x'))^2} - \frac{2}{3}\mathcal{A}_\alpha{}^\alpha(x')\frac{1}{(k^2+M(x'))^3} + \mathcal{O}\left(\frac{1}{k^8}\right).
\end{aligned} \tag{5.52}$$

If we then switch back to the coordinate representation, we can get rid of ∂^α by partially integrating, such that,

$$\begin{aligned}
\hat{\mathcal{G}}_F(y) = {}& \int\frac{\mathrm{d}^Dk}{(2\pi)^D}\left(\frac{e^{iky}}{k^2+M(x')} - \frac{1}{2}M^{(1)}_\alpha(x')y^\alpha\frac{e^{iky}}{(k^2+M(x'))^2} + \right.\\
&\left. - \frac{1}{3}\mathcal{A}_{\alpha\beta}(x')y^\alpha y^\beta\frac{e^{iky}}{(k^2+M(x'))^2} - \frac{2}{3}\mathcal{A}_\alpha{}^\alpha(x')\frac{e^{iky}}{(k^2+M(x'))^3}\right), \qquad (5.53)\\
\longrightarrow {}& \int\frac{\mathrm{d}^Dk}{(2\pi)^D}\left(\frac{1}{k^2+M(x')} - \frac{2}{3}\mathcal{A}_\alpha{}^\alpha(x')\frac{1}{(k^2+M(x'))^3}\right), y\to 0.
\end{aligned}$$

Using the integral representation,

$$\frac{1}{k^2+M(x')} = -i\int_0^\infty \mathrm{d}s\, e^{i(k^2+M(x')+i\epsilon)s},$$

we can rewrite,

$$\begin{aligned}\hat{\mathcal{G}}_F(y) &= -i\int_0^\infty \mathrm{d}s \int \frac{\mathrm{d}^D k}{(2\pi)^D} e^{i(k^2+M(x')+i\epsilon)s}\left(1-\frac{1}{9}\mathcal{A}_\alpha{}^\alpha(x')(is)^2\right),\, y\to 0\\ &\equiv -i\int_0^\infty \mathrm{d}s\, K(x;x';s)\,.\end{aligned} \tag{5.54}$$

It can be shown that, given such integral representation for the propagator, we can define the logarythm of the scalar Green function, which constitutes the effective action at one loop, as,

$$G_F(x;x') = \chi^{-1}(x;x')\hat{\mathcal{G}}_F(x;x') = -i\int_0^\infty \mathrm{d}s\, \chi^{-1}(x,x':\gamma)K(x;x';s) \tag{5.55}$$

$$\chi^{-1}(x,x':\gamma)K(x;x';s) = \langle x|\exp\left(g^{\frac{1}{4}}\left[\bar\nabla_\mu\bar\nabla^\mu+\alpha^2\bar R+i\epsilon\right]g^{-\frac{1}{4}}is\right)|x'\rangle\,;$$

$$\log G_F(x;x') = -i\int_0^\infty \mathrm{d}s\,\frac{\chi^{-1}(x,x':\gamma)K(x;x';s)}{is}\,. \tag{5.56}$$

Where in (5.55) we used that $G_F(x;x')$ is the inverse of the differential operator acting on the fields, and used the fact that $\chi(x',x') = -g(x') = 1$. Note that for this to be self consistent with conformal symmetry, we have to declare that $s\to\Omega^{-2}(x')s$, when conformal transformations are performed. In fact, this is not surprising, since the parameter s, note as Schwinger proper time, has the dimension of length squared. We can then choose its scaling to be at x or x', but it has to scale as a dimension 2 field.

We can proceed by performing the s integral in (5.54), to yield,

$$\hat{\mathcal{G}}_F(x,x') = \int_0^\infty \frac{\mathrm{d}s}{(4\pi is)^{D/2}}e^{-iM(x')s}\left(1-\frac{1}{9}\mathcal{A}_\alpha{}^\alpha(x')(is)^2\right),\, x\to x'\,. \tag{5.57}$$

We can finally write the one loop effective action as,

$$\begin{aligned}W_{eff} &= -\frac{i}{2}\mathrm{Tr}\log G_F = -\frac{i}{2}\int \mathrm{d}^D x\sqrt{-g}\lim_{x\to x'}\log G_F(x;x')\\ &= -\frac{1}{2}\int \mathrm{d}^D x\sqrt{-g}\int_0^\infty \mathrm{d}s\,\frac{K(x,x;s)}{is}\,.\end{aligned} \tag{5.58}$$

Performing the s integration, and keeping only the divergent terms will lead to [18],

$$\begin{aligned} W_{div} = &-\frac{i}{2}\int \frac{\mathrm{d}^D x}{(4\pi i)^{D/2}}\sqrt{-g}\Bigg((iM(x))^{\frac{D}{2}}\,\Gamma\left(-\frac{D}{2}\right) \\ &+\frac{1}{9}\mathcal{A}_{\alpha}{}^{\alpha}(x')\,(iM(x))^{\frac{D-4}{2}}\,\Gamma\left(2-\frac{D}{2}\right)\Bigg) = \\ &= -\frac{i}{2}\int \frac{\mathrm{d}^D x}{(4\pi)^{D/2}}\sqrt{-g}\Bigg[\left(\left(\xi-\frac{1}{6}\right)\bar{R}\right)^{\frac{D}{2}}\Gamma\left(-\frac{D}{2}\right) \\ &-\frac{1}{3}\left(\frac{1}{180}\bar{R}_{\alpha(\beta\gamma)\delta}\bar{R}^{\alpha(\beta\gamma)\delta}-\frac{1}{180}\bar{R}_{(\alpha\beta)}\bar{R}^{(\alpha\beta)}+\frac{(D-2)^2}{48}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}\right) \\ &\times\left(\left(\xi-\frac{1}{6}\right)\bar{R}\right)^{\frac{D-4}{2}}\Gamma\left(2-\frac{D}{2}\right)\Bigg]. \end{aligned} \tag{5.59}$$

Note that the divergent part of the 1-loop effective action (5.59) is Weyl invariant in any dimension [6], however the counterterms one adds to the theory require the introduction of the renormalization scale, and thus break the symmetry.

The action (5.59) is the 1-loop divergent action of the conformal theory we are studying, and still needs to be renormalized. This can be done by adding counterterms proportional to the pole $\frac{1}{D-4}$ and, in the spirit of a minimal subtraction scheme, only the finite part of the action (5.59) retained. In order to do this, we must introduce an arbitrary scale μ, which renders to expand $\left(\left(\xi-\frac{1}{6}\right)\bar{R}\right)^{\frac{D-4}{2}}$ and we leave discussions on the meaning of this choice to section 5.4.

[6] If the scalar field considered is charged under an internal symmetry group, such as the Higgs field is charged under $SU(2)$, this result should be accordingly changed, by an overall multiplicative factor counting the dimension of the representation in which the scalar field lives, and a correction proportional to the field strenght squared of the gauge connection. Since this only constitutes a complication for our purpose, we do not include it here. However, we will discuss it, as precisely this contribution induces the running of the gauge group coupling constant g.

To conclude this section, we give the renormalized action, following from Eq. (5.59),

$$W_{ren} = \frac{i}{2}\int \frac{\mathrm{d}^4x}{16\pi^2}\sqrt{-g}\Bigg[\left(\frac{3}{4}-\frac{\gamma_E}{2}-\frac{1}{2}\log\left(\frac{\left(\xi-\frac{1}{6}\right)\bar{R}}{4\pi\mu^2}\right)\right)\left(\left(\xi-\frac{1}{6}\right)\bar{R}\right)^2 + \frac{1}{3}\left(\frac{1}{180}\bar{R}_{\alpha(\beta\gamma)\delta}\bar{R}^{\alpha(\beta\gamma)\delta}-\frac{1}{180}\bar{R}_{(\alpha\beta)}\bar{R}^{(\alpha\beta)}+\frac{(D-2)^2}{48}\mathcal{T}_{\alpha\beta}\mathcal{T}^{\alpha\beta}\right) \times\left(\gamma_E+\log\left(\frac{\left(\xi-\frac{1}{6}\right)\bar{R}}{4\pi\mu^2}\right)\right)\Bigg], \tag{5.60}$$

where γ_E is the Euler-Mascheroni constant.

5.3.1 Boundary terms and local anomaly

There are terms that contribute to $\langle T^{\mu}_{\mu}\rangle$ as total derivatives. We refer to these contributions as the local anomaly, since as we shall see the energy momentum tensor they source is local and follows from varying a divergent but local action in the limit $D \to 4$. By contrast, the non-local terms produce generically a non local energy momentum tensor (even though its trace might become local, as is the case for example for the lagrangian $\frac{1}{\Box}R^2$), and can only be derived from a non local action, as shown in [24].

To expound on the meaning of these topological terms, let us consider the scalar 1-loop effective action (5.59) and rewrite the terms $\frac{1}{180}\bar{R}_{\alpha(\beta\gamma)\delta}\bar{R}^{\alpha(\beta\gamma)\delta} - \frac{1}{180}\bar{R}_{(\alpha\beta)}\bar{R}^{(\alpha\beta)} + \frac{(D-2)^2}{48}\mathcal{T}_{\alpha\beta}\mathcal{T}^{\alpha\beta}$, as a linear combination of the Weyl tensor, $C_{\alpha\beta\gamma\delta}$, the Gauss-Bonnet term (2.91) and $\mathcal{T}_{\alpha\beta}$,

$$W_{div} = \frac{1}{4\pi^2}\int \mathrm{d}^Dx\sqrt{-g}\Bigg[\Gamma\left(-\frac{D}{2}\right)\left(\left(\xi-\frac{1}{6}\right)\bar{R}\right)^{\frac{D}{2}} + \Gamma\left(2-\frac{D}{2}\right)\left(\alpha\mathcal{T}_{\alpha\beta}\mathcal{T}^{\alpha\beta}+\beta C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}+\gamma\mathcal{E}_4\right)\left(\left(\xi-\frac{1}{6}\right)\bar{R}\right)^{\frac{D-4}{2}}\Bigg], \tag{5.61}$$

where $\bar{R}$ is the Ricci scalar formed from the curvature tensor with torsion, $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor which is independent of the torsion trace,

$$\mathcal{E}_4 = \frac{1}{4!}\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\alpha\beta}{}^{\gamma\delta}\bar{R}^{\alpha}{}_{\gamma\mu\nu}\bar{R}^{\beta}{}_{\delta\lambda\sigma} \implies \frac{1}{D!}\epsilon^{\mu\nu\lambda\sigma\rho_1\cdots\rho_{D-4}}\epsilon_{\rho_1\cdots\rho_{D-4}\alpha\beta}{}^{\gamma\delta}\bar{R}^{\alpha}{}_{\gamma\mu\nu}\bar{R}^{\beta}{}_{\delta\lambda\sigma}, \tag{5.62}$$

is the Euler density which is in four dimensions a total divergence [44] and α, β and γ are numerical constants. The action (5.61) is divergent (in the sense that it yields divergent contributions to the Einstein's equation) and thus it ought to be renormalised. The first step in the renormalisation procedure is to identify the finite parts of the action.

To do that let us firstly analyse the contribution to the stress-energy tensor from the Euler density (5.62). Its variation gives a finite contribution to the stress-energy tensor and as such does not need any counter term. To see that let us vary the contribution of $\mathcal{E}_4$ to the effective action (5.61). We have,

$$\begin{aligned}&\frac{\delta}{\delta g^{\rho\tau}(z)}\int \mathrm{d}^D x\sqrt{-g}\epsilon^{\mu\nu\lambda\sigma\rho_1\cdots\rho_{D-4}}\epsilon_{\rho_1\cdots\rho_{D-4}\alpha\beta}{}^{\gamma\delta}\bar{R}^{\alpha}{}_{\gamma\mu\nu}\bar{R}^{\beta}{}_{\delta\lambda\sigma} \\ &\quad=\int \mathrm{d}^D x\sqrt{-g}\epsilon^{\mu\nu\lambda\sigma\rho_1\cdots\rho_{D-4}}\frac{\delta\left(\epsilon_{\rho_1\cdots\rho_{D-4}\alpha\beta}{}^{\gamma\delta}\right)}{\delta g^{\rho\tau}(z)}\bar{R}^{\alpha}{}_{\gamma\mu\nu}\bar{R}^{\beta}{}_{\delta\lambda\sigma}\,,\end{aligned} \tag{5.63}$$

where we dropped the following two terms,

$$\frac{\delta\left(\sqrt{-g}\epsilon^{\mu\nu\lambda\sigma\rho_1\cdots\rho_{D-4}}\right)}{\delta g^{\rho\tau}(z)}\quad\&\quad\sqrt{-g}\epsilon^{\mu\nu\lambda\sigma\rho_1\cdots\rho_{D-4}}\frac{\delta\left(\bar{R}^{\alpha}{}_{\gamma\mu\nu}\right)}{\delta g^{\rho\tau}(z)}\bar{R}^{\beta}{}_{\delta\lambda\sigma}\,,$$

the first one because the factors of $\sqrt{-g}$ cancel between $\sqrt{-g}$ and the Levi-Cività tensor and the second one because it vanishes due to the Bianchi identities. The term that is left in (5.63) is identically zero in $D = 4$ since in four dimensions $\epsilon_{\alpha\beta}{}^{\gamma\delta}$ yields contributions that are independent of the metric tensor. Taking account of this we finally arrive at the expression,

$$\begin{aligned}&\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\rho\tau}(z)}\int \mathrm{d}^D x\sqrt{-g}\epsilon^{\mu\nu\lambda\sigma\rho_1\cdots\rho_{D-4}}\epsilon_{\rho_1\cdots\rho_{D-4}\alpha\beta}{}^{\gamma\delta}\bar{R}^{\alpha}{}_{\gamma\mu\nu}\bar{R}^{\beta}{}_{\delta\lambda\sigma}\\ &=\frac{D-4}{D}g_{\rho\tau}\left(\epsilon^{\mu\nu\lambda\sigma}\epsilon_{\alpha\beta}{}^{\gamma\delta}\bar{R}^{\alpha}{}_{\gamma\mu\nu}\bar{R}^{\beta}{}_{\delta\lambda\sigma}\right)=\frac{D-4}{D}g_{\rho\tau}\mathcal{E}_4\,,\end{aligned}$$

which can be verified directly from (5.63) by evaluating

$$\frac{\delta\left(\epsilon_{\rho_1\cdots\rho_{D-4}\alpha\beta}{}^{\gamma\delta}\right)}{\delta g^{\rho\tau}(z)}\,.$$

This shows that we get a finite contribution to the stress-energy tensor from the divergent contribution proportional to the Euler density term in the effective action (5.61) and thus we do not have to add a counter term to renormalise it.

To be consistent, we should also check that the same term gives a finite contribution to the Weyl field source, D^μ. Indeed, upon noticing that $\mathcal{E}_4 = \bar{\nabla}_\mu\mathcal{V}^\mu$, where $\mathcal{V}^\mu$ has scaling dimension -4 under Weyl transformations, we can see that this is the case. Using the definition of the conformal derivative (2.46) one can show that in general D,

$$\bar{\nabla}_\mu\mathcal{V}^\mu=\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\mathcal{V}^\mu\right)-(D-4)T_\mu\mathcal{V}^\mu\,. \tag{5.64}$$

Which makes sense since the length dimension of $\mathcal{V}^\mu$ is -3 (it contains 3 derivatives acting on the metric), and thus $\int \mathrm{d}^D x \sqrt{-g} \bar{\nabla}_\mu \mathcal{V}^\mu$ is only dimensionless in $D = 4$. We can then conclude that, since the first term in Eq. (5.64) is a boundary term in any dimension, the Euler density contribution to the torsion source is,

$$\frac{1}{\sqrt{-g}} \frac{\delta}{\delta \mathcal{T}_\mu} \int \mathrm{d}^D x \sqrt{-g} \mathcal{E}_4 = -(D-4) \mathcal{V}^\mu \,, \tag{5.65}$$

which shows that the fundamental Ward identity (5.1) is in fact respected by this contribution. Note that this is not possible to achieve in a theory containing the metric only, since necessarily the Gauss-Bonnet contribution is finite and spoils the identity $\langle T^\mu_\mu \rangle = 0$.

This result is potentially important for the study of theories at the conformal point(s), defined by the vanishing of all beta functions of the theory's couplings λ_i, *i.e.* $\mu \frac{\partial \lambda_i}{\partial \mu} = 0$. Indeed, using Eq. (5.14), we can see that the only contribution to the trace anomaly at the conformal point(s) is the local anomaly (as all contributions from the dimensionful couplings are supposed to cancel there), which we just showed is not anomalous when a compensating torsion trace is introduced in the theory.

However, this is not yet the end of the story. In order to renormalise (5.61) one has to make all the terms (except possibly $\mathcal{E}_4$) finite in $D = 4$ and the only way of doing that within dimensional regularisation is to add scale dependent counterterms that in this way introduce a scale dependence in the renormalized action, thus breaking Weyl symmetry. We will now discuss this point, and how one can choose a renormalization scheme that respects the Ward identities.

5.4 The renormalization scale Λ

As it is practice in the study of quantum field theory, in order to extract the finite part of (5.59) we must introduce a dimensionful parameter, the so-called renormalization scale μ, that allows us to consistently expand

$$\left(\left(\xi - \frac{1}{6} \right) \bar{R} \right)^{\frac{D-4}{2}} \,.$$

Usually this does not pose a problem, however, in the context of the geometric theory we have discussed so far, it is not so obvious what exactly we mean by "μ". In particular, there is no invariant way of choosing a "constant" mass scale μ to renormalize the theory.

This reflects the fact that, in a strictly Weyl invariant theory, there is no universal notion of scale. Observers are then forced to measure some dimensionfull quantity and use it as a reference for any further measurement. There is however no guarantee that such a measurement performed at a different space-time point would yield the same result.

This is because in the geometry studied in this thesis, parallel transport of dimensionfull quantities should be defined using the derivative $\overline{\nabla}$, rather than the usual covariant derivative. Then, suppose two observers $\mathcal{O}_1, \mathcal{O}_2$ agree on the definition of the scale M at some space-time point x_0, but follow two different trajectories γ_1, γ_2 and then meet again at the space-time point x_1. Then, the scale they will measure at x_1 are given by,

$$\begin{aligned} M_1(\mathcal{O}_1) =& M \exp\left(\Delta_M \int_{\gamma_1} \mathrm{d}x^\mu \mathcal{T}_\mu\right) \implies \\ \frac{M(\mathcal{O}_1)}{M(\mathcal{O}_2)} =& M \exp\left(\Delta_M \int_{\gamma_1 \cup \gamma_2} \mathrm{d}x^\mu \mathrm{d}x^\nu \mathcal{T}_{\mu\nu}\right) , \end{aligned} \tag{5.66}$$

where Δ_M is the scaling dimension of M, since the parallel transport equation reads,

$$\dot{\gamma}^\mu_{1/2} \partial_\mu M + \Delta_M \dot{\gamma}^\mu_{1/2} \mathcal{T}_\mu M = 0 .$$

The right hand side of (5.66) shows that observers using the Weyl field to compare measurements will measure a discrepancy in their definition of scale which is given by the flux of the transverse part of the torsion trace through the surface $\gamma_1 \cup \gamma_2$. This makes it impossible to define a notion of constant scale in the general case of non vanishing torsion flux (5.66).

The best definition we can achieve is therefore,

$$\theta = \frac{1}{\overset{\circ}{\Box}} \overset{\circ}{\nabla}{}^\mu T_\mu . \tag{5.67}$$

Eq. (5.67) is a non local definition, and requires the appropriate boundary conditions to be evaluated. However, it has the advantage that the field,

$$\mu(x) = \exp\left(-\theta(x)\right) , \tag{5.68}$$

can be used as a geometrically defined renormalization scale. Such a procedure is analogous to that defined in [55, 112, 113], and one might wonder whether it is justified physically. In fact, in order to renormalize the theory this way we must introduce the non local operator,

$$\frac{\left(\mu(x)\right)^{D-4}}{D-4} ,$$

which contains, away from $D = 4$, non integer powers of the field. In this sense it is a non local counterterm, which violates one of the fundamental principles of dimensional regularization. In fact, there are many choices for $\mu(x)$, all of which invoke a physical scalar condensate in the effective action. Moreover, if $\mu(x)$ is a field, even though non dynamical, it must generate a constrain, which is bad news for any renormalization scheme. It is of course unclear whether this procedure is the correct one to use, as it can introduce an unwanted, unphysical dependence in the effective action.

If more fields are present, for example, one encounters an ambiguity: the condensate of fields is supposed to minimize the effective potential, which to be computed requires the introduction of the renormalization scale. It is not difficult to understand that choosing a different condensate to renormalise the theory gives a different result for the effective potential, which in turn can change the value that minize the effective potential.

On the other hand the effect (5.66) might be regarded as a physical effect, and we should not try to construct scale dependent renormalization schemes to try and respect the Ward identities (5.12). Instead, we can try to fix the identities (5.12) such that they are respected in dimensional regularization. In order to do so, we analyse the behaviour of the theory around $D = 4$, treating $D - 4$ as a small parameter, in a way that does not modifies the Feynman rules.

Consider, to this end, the theory,

$$S = \int \mathrm{d}^D x \sqrt{-g}\Bigg[-\frac{1}{2}\left(\partial_\mu \phi + \frac{D-2}{2}\mathcal{T}_\mu \phi\right)\left(\partial^\mu \phi + \frac{D-2}{2}\mathcal{T}^\mu \phi\right) \\ -\frac{1}{2}m^2\phi^2 + \frac{\xi}{2}\phi^2 R - \frac{\lambda}{4}\phi^4\Bigg], \tag{5.69}$$

its Ward identities in dimensional regularization are given by,

$$\langle \nabla_\mu D^\mu + T^\mu_\mu \rangle = m^2 \langle \phi^2 \rangle + \frac{(D-4)}{4}\lambda \langle \phi^4 \rangle . \tag{5.70}$$

Taking seriously the right hand side of (5.70), we pick up finite contributions, whenever the prefactor $D - 4$ hits one of the divergences of the four points interaction on the right hand side.

This is the consequence of defining the theory in the neighbourhood of $D = 4$, and retaining the coefficients of the expansion that are finite in the limit $D \to 4$ will simply lead to the modification (5.14), where the right hand side, given by the beta functions of the theory, can be calculated using the prescrition in (5.70).

We conjecture that the same should be true for the gravitational anomaly: if the anomalous scaling is accounted for in the derivation of the Ward identities, no further terms should arise. The anomalous trace that is generated in the energy momentum trace, is compensated for by the quantum corrections to the dilatation current, which can be consistently computed in perturbative dimensional regularization.

Stated otherwise, this hypothesis means that the two quantities, $\frac{\delta S}{\delta g^{\mu\nu}}$ and $\frac{\delta S}{\delta \mathcal{T}_\mu}$ are not indipendent from each others, but are linked by the Weyl gauge symmetry. Therefore their anomalous dimensions must conspire in such a way to cancel at all loops.

To provide evidence in favour of this conjecture, in next section we set up a semi-classical computation, in which $g_{\mu\nu}$ and $\mathcal{T}_\mu$ are treated as external sources for a quantum scalar field, such as the one defined in (3.6), and compute the lowest order corrections to the vertices $\Delta^{\alpha\beta}_{\mathcal{T}\mathcal{T}}, \Delta^{\mu\nu\beta}_{gT}$. These vertices correspond to the first coefficients in the expansion of the energy momentum tensor and dilation current, when the external fields are small.

The general expression for the Weyl anomaly, as can be read for example in [18] is, in the massless limit,

$$\lim_{m\to 0}\nabla_\mu\langle D^\mu\rangle + \langle T^\mu_\mu\rangle = \\ = -\frac{1}{16\pi^2}\left(\frac{1}{120}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} - \frac{1}{360}\mathcal{E}_4 + \frac{1}{180}\Box R\right) . \tag{5.71}$$

Taking one functional derivative of this relation, and setting $g_{\mu\nu}\to\eta_{\mu\nu}\,,\mathcal{T}_\alpha\to 0$ afterwards, we obtain the Ward identities obeyed by the correlators $\Gamma_{g\mathcal{T}}$, $\Gamma_{\mathcal{T}\mathcal{T}}$ (see below the precise definition, in (5.72)). This singles out the contributions on the right hand side of (5.71) that are linear in $g_{\mu\nu}\,,\mathcal{T}_\alpha$, which are contained in the term $\frac{1}{180}\Box R$. Since we do not see these structure in the correlators $\Gamma_{g\mathcal{T}}\,,\Gamma_{\mathcal{T}\mathcal{T}}$ we conclude that no term $\frac{1}{180}\Box R$ should be written on the right hand side of (5.71).

A disclaimer is however required: it is still the case that $\langle T^\mu_\mu\rangle$ acquires anomalous contributions, but so does $\langle D^\mu\rangle$, in precisely the same way. We thus still have a quantum breaking of scale invariance, which is in fact a global symmetry of the theory, but the anomalous trace that is generated like so, is compensated by an anomalous contribution in the dilatation current (which should in fact be conserved, *i.e.* $\nabla_\mu D^\mu = 0$, for a scale invariant theory [109]).

In fact Eq. (5.71) seems to only take into account the contributions to the identity coming from the energy momentum tensor, and not accounting for the quantum corrections acquired, at one loop, by the dilatation current. These two must balance, since the theory (5.73) is Weyl invariant in any dimensions (unlike the same theory with the addition of the interaction $\lambda\phi^4$).

This constitutes but the first step in understanding the quantum behaviour of the theory, as to establish if (5.73) is a self consistent theory, whose quantum breaking of scaling symmetry is well behaved, we should evaluate also the quadratic contributions to the right hand side of (5.71), which require computing the 3-point functions. We leave this computation for future work.

5.5 Ward identities

In this section we are going to analyse the one loop quantum corrections to the dilatation current, given by,

$$D^\mu(x) = D^\mu_0(x) + \int \mathrm{d}^D y\sqrt{-g}\,\Gamma^{\mu\alpha\beta}_{\mathcal{T}g}(x,y)\delta g^{\alpha\beta}(y) \\ + \int \mathrm{d}^D y\sqrt{-g}\,\Gamma^{\mu\alpha}_{\mathcal{T}\mathcal{T}}(x,y)\delta T_\alpha(y)\,, \tag{5.72}$$

where D_0 is the tree level dilatation current, in the theory defined by the following action,

$$S = \int \mathrm{d}^D x \sqrt{-g} \Bigg[-\frac{1}{2} \left(\partial_\mu \phi + \frac{D-2}{2} \mathcal{T}_\mu \phi \right) \left(\partial^\mu \phi + \frac{D-2}{2} \mathcal{T}^\mu \phi \right) \\ - \frac{1}{2} m^2 \phi^2 + \frac{\xi}{2} \phi^2 R \Bigg] . \tag{5.73}$$

In Eq. (5.73) we have added a mass term, which explicitely breaks Weyl symmetry, for two reasons: first, to avoid unpleasant infrared divergencies in the scalar loop integrals, and secondly, to analyse in what way an explicit breaking of Weyl symmetry will affect the Ward-Takahashi identities for Weyl symmetry. We will see that this explicit symmetry breaking term introduces a mass for the Goldstone modes, which vanishes when the limit $m \to 0$ is taken.

We will show that the identities obeyed by the two points functions (5.72), are anomalous for $\xi = -\frac{1}{6}$. More precisely, in this case $D^\mu \propto (D-4)$, such that finite loop contributions are generated from the divergent part of the gT correlator. This anomaly does not violate the extended Ward identities where the torsion field is included, but would violate the trace identities in a theory with the metric only.

This illustrates how the introduction of a torsion field is necessary to avoid the appearence of anomalies. In [25, 114] the authors argue that the Weyl anomaly in quantum field theory signals the presence of massless scalar degrees of freedom, which are unaccounted for in the original theory. They can be introduced to localise the anomaly, as explained for example in [24], such that the Ward identities with the inclusion of these fields are obeyed.

Given the results of this section, we can claim we have evidence of the existence of a self consistent framework, in which such degrees of freedom are attributed to the torsion trace and they obey local equations of motion. In this theory, one does not have to construct by hand an action to localize the anomaly, since the required degrees of freedom are included to begin with, and therefore physical. Some of these correspond to Goldstone modes of broken scale transformations, and are supposed to be massless, *i.e.* their mass is protected by the scaling symmetry. In what follows we will illustrate this more precisely.

The action (5.73) is invariant under the combined transformation,

$$\delta_\theta : \quad g_{\mu\nu} \to (1 + 2\theta(x)) g_{\mu\nu} \, , T_\mu \to T_\mu + \partial_\mu \theta \, , \\ \phi \to \left(1 - \frac{D-2}{2} \theta(x) \right) \phi \, , \tag{5.74}$$

$$\delta_\xi : \quad g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu} \, , , T_\mu \to T_\mu + \mathcal{L}_\xi T_\mu \, , \phi \to \phi + \xi^\mu \partial_\mu \phi \, , \tag{5.75}$$

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda} \partial_\nu \xi^\lambda + g_{\nu\lambda} \partial_\mu \xi^\lambda = 2 \overset{\circ}{\nabla}_{(\mu} \xi_{\nu)} \, , \\ \mathcal{L}_\xi T_\mu = \xi^\lambda \partial_\lambda T_\mu + T_\lambda \partial_\mu \xi^\lambda \, .$$

These transformation satisfy the algebra,

$$[\delta_\xi,\delta_\theta]g_{\mu\nu} = 2(\xi^\lambda\partial_\lambda\theta)g_{\mu\nu}\,,\ [\delta_\xi,\delta_\theta]\phi = -\frac{D-2}{2}(\xi^\lambda\partial_\lambda\theta)\phi\,,\ [\delta_\xi,\delta_\theta]T_\mu = 0\,,$$
$$[\delta_{\theta_1},\delta_{\theta_2}]\psi_I = 0\,, [\delta_{\xi_1},\delta_{\xi_2}]\psi_I = \mathcal{L}_{[\xi_1,\xi_2]}\psi_I\,, \tag{5.76}$$

where ψ_I denotes an arbitrary representation of the Lorentz group. Defining the vacuum vacuum transition amplitude for the action S_ϕ (5.73), as

$$iW = \log\langle in|out\rangle = \log\left(\int \overline{\mathcal{D}}\phi e^{iS_\phi}\right), \tag{5.77}$$

we have,

$$\begin{aligned}\langle D^\mu\rangle &= \frac{1}{\sqrt{-g}}\frac{\delta W}{\delta T_\mu} = \\ = -&\left(\frac{D-2}{2}+2\xi(D-1)\right)\left\langle \mathbb{T}^*\left\{\phi(x)\left(\partial^\mu+\frac{D-2}{2}\mathcal{T}^\mu\right)\phi(x)\right\}\right\rangle, \end{aligned}\tag{5.78}$$

$$\begin{aligned}\langle T^{\mu\nu}\rangle &= -\frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g_{\mu\nu}} = \\ &=\left\langle \mathbb{T}^*\left\{-\nabla^\mu\phi\nabla^\nu\phi+g^{\mu\nu}\left(\frac{1}{2}\nabla_\lambda\phi\nabla^\lambda\phi+\frac{m^2}{2}\phi^2\right)\right.\right. \\ &\left.\left.+\xi\left(R^{(\mu\nu)}-\frac{g^{\mu\nu}}{2}R+g^{\mu\nu}\nabla^2-\nabla^{(\mu}\nabla^{\nu)}\right)\phi^2\right\}\right\rangle, \end{aligned}\tag{5.79}$$

where $\langle\mathbb{T}^*\cdot\rangle = \frac{\int\mathcal{D}\phi(\cdot)e^{iS}}{\int\mathcal{D}\phi e^{iS}}$, denotes the $\mathbb{T}^*$ ordered expectation value of operators, namely the time ordered product with the derivatives not acting on the θ functions that time ordering requires. We want to verify the Ward identities,

$$\nabla_\mu\langle D^\mu\rangle + \langle T^\mu_\mu\rangle = \frac{m^2}{2}\langle\phi^2\rangle\,, \tag{5.80}$$
$$\nabla_\mu\langle T^\mu_\nu\rangle + \mathcal{T}_\nu\nabla_\mu\langle D^\mu\rangle = \mathcal{T}_{\mu\nu}\langle D^\mu\rangle\,, \mathcal{T}_{\mu\nu} = \partial_\mu\mathcal{T}_\nu - \partial_\nu\mathcal{T}_\mu\,, \tag{5.81}$$

in perturbative dimensional regularisation. Notice how the right hand side of Eq. (5.80) acquires the contribution $\propto m^2$. Such a contribution, present at the classical and at the quantum level, simply comes from the explicit symmetry breaking term in (5.73)

The first crucial object we need is the two points functions,

$$\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}(x,y) = \frac{\delta\langle D^\alpha(x)\rangle}{\delta\mathcal{T}_\beta(y)}\,,\ \Gamma^{\mu\nu\beta}_{g\mathcal{T}}(x,y) = \frac{\delta\langle T^{\mu\nu}(x)\rangle}{\delta\mathcal{T}_\beta(y)} = \frac{\delta\langle D^\beta(y)\rangle}{\delta g_{\mu\nu}(x)}\,. \tag{5.82}$$

From the theory defined by (5.73) we find, in the limit $\mathcal{T}_\mu\to 0\,, g_{\mu\nu}\to\eta_{\mu\nu}$

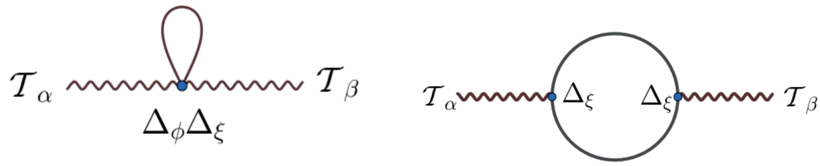


FIGURE 5.1: Here the Feynman diagrams contributing to the $\mathcal{T}\mathcal{T}$ 2 points vertex. The couping strengths are defined as, $\Delta_\xi \equiv (\frac{D-2}{2} + 2\xi(D-1))$, and $\Delta_\phi = -\frac{D-2}{2}$. The external legs are attached at the points x and x', which explains why we refer to the left diagram as local (or $\propto \delta^D(x-x')$), and the one to the right as non-local.

$$\begin{aligned}\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}(x;y)\Big|_{\mathcal{T}=0\, g=\eta} &= \left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \langle \mathbb{T}^* \left\{\phi\partial^\alpha_x\phi\phi\partial^\beta_y\phi\right\}\rangle_C \\ &- \frac{D-2}{2}\left(\frac{D-2}{2} + 2\xi(D-1)\right)\eta^{\alpha\beta}\delta^D(x-y)\langle\mathbb{T}^*\{\phi(x)\phi(y)\}\rangle_C\,,\end{aligned} \tag{5.83}$$

where $\langle\cdot\rangle_C$ indicates that only the connected graphs contribute to the expectation value.

Similarly, we can compute $\Gamma^{\mu\nu\alpha}_{gT}(x;y)$, to find,

$$\begin{aligned}\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(x;y)\Big|_{\tau=0\, g=\eta} &= \left\langle \mathbb{T}^* \left\{ \left(\eta^{\mu\nu}\delta^D(x-y)\phi\partial^\alpha_x\phi - 2\eta^{\alpha(\mu}\delta^D(x-y)\phi\partial^{\nu)}_x\phi\right)\right\}\right\rangle_C \\ &- (D-1)\xi\delta^D(x-y)\langle\mathbb{T}^*\left\{\left(2\eta^{\alpha(\mu}\partial^{\nu)}_x\phi^2 - \eta^{\mu\nu}\partial^\alpha_x\phi^2\right)\right\}\rangle_C \\ &- \left(\frac{D-2}{2} + 2\xi(D-1)\right)\left\langle\mathbb{T}^*\left\{\left(-\partial^\mu_x\phi\partial^\nu_x\phi + \eta^{\mu\nu}\left(\frac{1}{2}\partial^x_\lambda\phi\partial^\lambda_x\phi + \frac{m^2}{2}\phi^2\right)\right.\right.\right. \\ &\left.\left.\left. + \xi\left(\eta^{\mu\nu}\partial^x_\lambda\partial^\lambda_x - \partial^{(\mu}_x\partial^{\nu)}_x\right)\phi^2\right)\phi\partial^\alpha_y\phi\right\}\right\rangle_C\,.\end{aligned} \tag{5.84}$$

After Wick contracting, and using that

$$\begin{aligned}&\langle\mathbb{T}^*\left(\phi(x)\phi(y)\right)\rangle \equiv i\Delta(x,y) = i\Delta(x-y)\,, \\ &\left(\partial^2 - m^2\right)i\Delta(x,y) = \delta^D(x-y)\end{aligned} \tag{5.85}$$

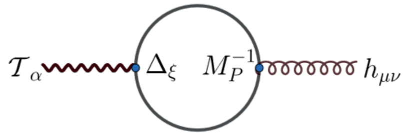


FIGURE 5.2: The Feynman diagram contributing to the $g\mathcal{T}$ 2-point vertex. The couping strengths are defined as, $\Delta_\xi \equiv (\frac{D-2}{2} + 2\xi(D-1))$, and M_P^{-1} being the Planck mass, the coupling strength of the physical (*i.e.* dimensionful) graviton. Again the external legs are attached to the points x and x' (5.82).

which implies that $\partial_x \Delta(x,y) = -\partial_y \Delta(x,y)$, and suppressing the subcript $\Big|_{\tau=0\, g=\eta}$ for the sake of notation and simplicity, we arrive at,

$$\begin{aligned}\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(x;y) =& \delta^D(x-y)\Big[-\eta^{\mu\nu}\partial_y^\alpha\, i\Delta(x,y) + 2\eta^{\alpha(\mu}\partial_y^{\nu)}\, i\Delta(x,y)\\ &+ 2\xi(D-1)(2\eta^{\alpha(\mu}\partial_y^{\nu)}\, i\Delta(x,y) - \eta^{\mu\nu}\partial_y^\alpha\, i\Delta(x,y)\Big]\\ -&\left(\frac{D-2}{2} + 2\xi(D-1)\right)\Bigg[\partial_x^\mu\, i\Delta(x,y)\partial_x^\nu\partial_x^\alpha\, i\Delta(x,y) + \partial_x^\nu\, i\Delta(x,y)\partial_x^\mu\partial_x^\alpha\, i\Delta(x,y)\\ &- \eta^{\mu\nu}\Big(\partial^x_\lambda\, i\Delta(x,y)\partial_x^\lambda\partial_x^\alpha\, i\Delta(x,y) + m^2\, i\Delta(x,y)\partial_x^\alpha\, i\Delta(x,y)\Big)\\ &+ 2\xi(\eta^{\mu\nu}\partial_x^2 - \partial_x^\mu\partial_x^\nu)\, i\Delta(x,y)\partial_x^\alpha\, i\Delta(x,y)\Bigg].\end{aligned} \tag{5.86}$$

and

$$\begin{aligned}\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}(x;y) =& -\left(\frac{D-2}{2} + 2\xi(D-1)\right)^2\Bigg(i\Delta(x;y)\partial^\alpha\partial^\beta\, i\Delta(x;y)\\ &+ \partial^\alpha\, i\Delta(x;y)\partial^\beta\Delta(x;y)\Bigg)\\ &- \frac{D-2}{2}\left(\frac{D-2}{2} + 2\xi(D-1)\right)\eta^{\alpha\beta}\delta^D(x-y)\, i\Delta(x;y).\end{aligned} \tag{5.87}$$

The Feynman diagram contributions to the vertices (5.86–5.87) are represented in figures 5.1 and 5.2. Since the coupling strength of the interactions is $\Delta_\xi = 1 + 6\xi + \mathcal{O}(D-4)$, this perturbation theory can be trusted when $\xi \simeq -\frac{1}{6}$, that is near its conformal value.

When evaluated in momentum space, Eqs. (5.86–5.87) becomes straightforwardly,

$$\Gamma^{\mu\nu\alpha}_{gT}(p) = i\int \frac{\mathrm{d}^D l}{(2\pi)^D}\Big[\Big(\eta^{\mu\nu}l^\alpha - 2\eta^{\alpha(\mu}l^{\nu)} - 2\xi(D-1)(2\eta^{\alpha(\mu}l^{\nu)}\,i - \eta^{\mu\nu}l^\alpha\Big)\; i\Delta(l)\Big]$$
$$+ i\int \frac{\mathrm{d}^D l}{(2\pi)^D}\left(\frac{D-2}{2} + 2\xi(D-1)\right)\Big[(p^\mu - l^\mu)l^\nu l^\alpha + (p^\nu - l^\nu)l^\mu l^\alpha$$
$$- \eta^{\mu\nu}\Big((p\cdot l - l^2)l^\alpha + m^2 l^\alpha\Big) + 2\xi(\eta^{\mu\nu}p^2 - p^\mu p^\nu)l^\alpha\Big]\; i\Delta(p-l)\; i\Delta(l)\,. \tag{5.88}$$

and,

$$\Gamma^{\alpha\beta}_{TT}(p) = \left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \int \frac{\mathrm{d}^D l}{(2\pi)^D}\Big(p^\alpha p^\beta - l^\alpha p^\beta\Big)\; i\Delta(l)\, i\Delta(p-l)$$
$$- \frac{D-2}{2}\left(\frac{D-2}{2} + 2\xi(D-1)\right)\eta^{\alpha\beta}\int \frac{\mathrm{d}^D l}{(2\pi)^D}\, i\Delta(l) \tag{5.89}$$

The Ward identities that these object must satisfy are,

$$\partial_\alpha \Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}(x,y) + \eta_{\mu\nu}\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(x,y) = \tag{5.90}$$
$$= \; -\left(\frac{D-2}{2} + 2\xi(D-1)\right)\Big[m^2\Big(2\, i\Delta(x,y)\partial^\alpha_y\, i\Delta(x,y)\Big)\Big]$$
$$\partial_\mu \Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(x,y) + \delta^D(x-y)\eta^{\nu\alpha}\partial^x_\mu \langle D^\mu(x)\rangle = \tag{5.91}$$
$$= \; \langle D^\mu(x)\rangle\Big(\eta^{\alpha\nu}\partial^x_\mu - \delta^\alpha_\mu\partial^\nu_x\Big)\,\delta^D(x-y)\,.$$

where we have taken the limit $g_{\mu\nu} \to \eta_{\mu\nu}$, and $\mathcal{T}_\nu \to 0$.

In order to evaluate the contact terms, we will use the point-splitting tecnique, such that for example,

$$\langle\phi(x)\partial^\mu_x\phi(x)\rangle = \lim_{x\to x'}\langle\phi(x')\partial^\mu_x\phi(x)\rangle = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\,(ik^\mu)\; i\Delta(k)\,,$$
$$\partial_\mu\langle\phi(x)\partial^\mu_x\phi(x)\rangle = \lim_{x\to x'}(\partial^x_\mu + \partial^{x'}_\mu)\langle\phi(x')\partial^\mu_x\phi(x)\rangle = 0\,, \tag{5.92}$$

such that in momentum space (5.91) becomes, owing to the fact that $\langle D^\mu\rangle = \langle\partial_\mu D^\mu\rangle = 0$,

$$p_\mu \Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(p) = 0\,. \tag{5.93}$$

In a similar fashion, we can obtain the momentum representation of (5.90),

$$p_\alpha \Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}(p) - i\eta^{\mu\nu}\Gamma^{g\mathcal{T}}_{\mu\nu\alpha}(p) = \\ = -2\left(\frac{D-2}{2} + 2\xi(D-1)\right) m^2 \left(\int \frac{\mathrm{d}^D l}{(2\pi)^D} l_\alpha \, i\Delta(l)\, i\Delta(p-l)\right) . \tag{5.94}$$

5.5.1 Tensorial reduction

Define the tensorial integrals,

$$A(p) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{((p-l)^2 + m^2)} \tag{5.95}$$

$$A^\mu(p) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu}{((p-l)^2 + m^2)} \tag{5.96}$$

$$A^{\mu\nu}(p) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu l^\nu}{((p-l)^2 + m^2)} \tag{5.97}$$

$$A^{\mu\nu\alpha}(p) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu l^\nu l^\alpha}{((p-l)^2 + m^2)} , \tag{5.98}$$

and,

$$B_0(p) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{(l^2 + m^2)((p-l)^2 + m^2)} \tag{5.99}$$

$$B^\mu(p) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu}{(l^2 + m^2)((p-l)^2 + m^2)} \tag{5.100}$$

$$B^{\mu\nu}(p) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu l^\nu}{(l^2 + m^2)((p-l)^2 + m^2)} \tag{5.101}$$

$$B^{\mu\nu\alpha}(p) = \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu l^\nu l^\alpha}{(l^2 + m^2)((p-l)^2 + m^2)} , \tag{5.102}$$

in terms of which it is straightforward to write our $g\mathcal{T}$ 2-point function (5.86) as,

$$\Gamma^{gT}_{\mu\nu\alpha} = i\left(\frac{D-2}{2} + 2\xi(D-1)\right)\Big[p_\mu B_{\nu\alpha}(p) + p_\nu B_{\mu\alpha}(p) - 2B_{\mu\nu\alpha}(p) \\ - g_{\mu\nu}\left(p^\sigma B_{\sigma\alpha}(p) - g^{\lambda\sigma}B_{\lambda\sigma\alpha}(p)\right) - m^2 g_{\mu\nu} B_\alpha(p) - 2\xi\Big(p^2 g_{\mu\nu} \\ - p_\mu p_\nu\Big) B_\alpha(p)\Big] , \tag{5.103}$$

First, we can now evaluate $p^\mu \Delta^{gT}_{\mu\nu\alpha}$, to check that the Ward identity for diffeomorphsms is satisfied,

$$\begin{aligned}
p^\mu \Gamma^{gT}_{\mu\nu\alpha} =& i\left(\frac{D-2}{2}+2\xi(D-1)\right)\Big[p^2 B_{\nu\alpha}(p)+p_\nu p^\mu B_{\mu\alpha}(p) \\
&-2p^\mu B_{\mu\nu\alpha}(p)-p_\nu\left(p^\sigma B_{\sigma\alpha}(p)-g^{\lambda\sigma}B_{\lambda\sigma\alpha}(p)\right)-m^2 p_\nu B_\alpha(p)\Big]= \\
=& i\left(\frac{D-2}{2}+2\xi(D-1)\right)\Big[A_{\nu\alpha}(0)-A_{\nu\alpha}(p)+p_\nu A_\alpha(p)\Big] \\
=& 0\,,\forall\ D\,,
\end{aligned} \tag{5.104}$$

as one can check by using the reduction formulas,

$$\begin{aligned}
p^\mu B_{\mu\alpha}(p) &= \frac{1}{2}\left(p^2 B_\alpha(p)+A_\alpha(p)-A_\alpha(0)\right)\,, \\
g^{\lambda\sigma}B_{\lambda\sigma\alpha}(p) &= A_\alpha(p)-m^2 B_\alpha(p)\,, \\
B_\alpha(p) &= \frac{1}{2p^2}p_\alpha\left(p^2 B(p)+A(p)-A(0)\right)\,,
\end{aligned} \tag{5.105}$$

and the expressions (5.119–5.120) in appendix 5.7. The formulas (5.105) are based on the Passarino-Veltman reduction formula, which simply rewrites, $p\cdot l=\frac{1}{2}\left(p^2+l^2-2(p-l)^2\right)$, inside the loop integrals, to reduce the tensorial loop integrals to scalar ones.

Next, given that the dilatation current 2-point function in momentum space becomes (5.88), we have,

$$\begin{aligned}
\Gamma^{\alpha\beta}_{TT}(p) =& \left(\frac{D-2}{2}+2\xi(D-1)\right)^2\left(p^\alpha p^\beta B(p)-B^\alpha(p)p^\beta\right) \\
&-\frac{D-2}{2}\left(\frac{D-2}{2}+2\xi(D-1)\right)\eta^{\alpha\beta}A(0)\,.
\end{aligned} \tag{5.106}$$

Next, using the reduction formulas (5.105) to infer that,

$$\begin{aligned}
\Gamma^{\alpha\beta}_{TT}(p) =& \left(\frac{D-2}{2}+2\xi(D-1)\right)^2\left(p^2 B(p)-A(p)\right)\frac{p^\alpha p^\beta}{2p^2} \\
&+\left(\frac{D-2}{2}+2\xi(D-1)\right)\left(\frac{1}{2}\left(\frac{D-2}{2}+2\xi(D-1)\right)p^\alpha p^\beta\right. \\
&\left.-\frac{D-2}{2}p^2\eta^{\alpha\beta}\right)\frac{A(0)}{p^2}\,,
\end{aligned} \tag{5.107}$$

allows us to compute its divergence,

$$p_\alpha \Gamma^{\alpha\beta}_{TT}(p) = \left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 \left(p^2 B(p) - A(p) + A(0)\right) \frac{p^\beta}{2} - \frac{D-2}{2}\left(2\xi(D-1) + \frac{D-2}{2}\right) p^\beta A(0)\,. \tag{5.108}$$

Finally taking a trace of the metric torsion 2-point function (5.103) gives,

$$\begin{aligned}\eta^{\mu\nu}\Gamma^{gT}_{\mu\nu\alpha} =& i\left(\frac{D-2}{2} + 2\xi(D-1)\right)\Big[(2-D)\left(p^\lambda B_{\lambda\alpha}(p) - \eta^{\mu\nu}B_{\mu\nu\alpha}(p)\right) \\ &- Dm^2 B_\alpha(p) - 2\xi\,(D-1)\,p^2 B_\alpha(p)\Big]\,. \\ =& -i\left(\frac{D-2}{2} + 2\xi(D-1)\right)^2 p^2 B_\alpha(p) + \left(\frac{D-2}{2} + 2\xi(D-1)\right)\times \\ &\times\left(\frac{D-2}{2}A_\alpha(p) - 2m^2 B_\alpha(p)\right),\end{aligned} \tag{5.109}$$

where we used that $A_\alpha(0) = 0$. We can finally evaluate the sum (5.108–5.109), to find,

$$p_\alpha \Gamma^{\alpha\beta}_{TT}(p) - i\eta^{\mu\nu}\Gamma^{gT}_{\mu\nu\alpha} = -2\left(\frac{D-2}{2} + 2\xi(D-1)\right) m^2 B_\alpha(p)\,, \tag{5.110}$$

where we used once more the reduction, $B^\alpha(p) = \frac{1}{2p^2}(p^2 B(p) + A(p) - A(0))p^\alpha$ in dimensional regularisation, and that to this order in perturbation theory, $A(0) - A(p) = 0$. We point out that the integrals computed here and in the appendix are free of infra-red divergencies, as one can see by taking limits $m \to 0$ in the expressions we report. In fact, all infra-red divergencies are regulated by the addition of the mass term, and if the limit $m \to 0$ is well defined, no physically relevant process will be subtracted, in the infra-red, by dimensional regularization.

To end this section, we report the regularized expression for $\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}(p)$ and $\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(p)$. The two dilatation currents 2-points function, reads,

$$\begin{aligned}
\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T},\text{ren}}(p) =& \frac{1}{2}\left(\frac{D-2}{2}+2\xi(D-1)\right)^2\int\frac{\mathrm{d}^D l}{(2\pi)^D}\left(p^\alpha p^\beta\right)\Delta(l)\Delta(p-l)\\
&-\frac{D-2}{2}\left(\frac{D-2}{2}+2\xi(D-1)\right)\eta^{\alpha\beta}\int\frac{\mathrm{d}^D l}{(2\pi)^D}\Delta(l) =\\
=&\frac{i}{16\pi^2}\frac{p^\alpha p^\beta}{2}\Bigg((6\xi+1)^2\left(\log(4\pi)-\gamma_E+\log\left(\frac{\mu^2}{m^2}\right)\right)+4\xi(6\xi+1)\\
&+\frac{2m^2\left(4\xi+1-(6\xi+1)\left(\log(4\pi)-\gamma_E+\log\left(\frac{\mu^2}{m^2}\right)\right)\right)}{p^2}\\
&-\frac{(6\xi+1)^2}{p^2}\sqrt{p^2\left(p^2+4m^2\right)}\log\left(\frac{\sqrt{p^2\left(p^2+4m^2\right)}+2m^2+p^2}{2m^2}\right)\Bigg)+\\
&+m^2\left((6\xi+1)\left(\log(4\pi)-\gamma_E+\log\left(\frac{\mu^2}{m^2}\right)\right)+4\xi+1\right)\times\\
&\times\left(g^{\alpha\beta}-\frac{p^\alpha p^\beta}{p^2}\right).
\end{aligned} \tag{5.111}$$

Note that the finite terms in this computation are computed by keeping all the finite contributions in the expansion around $D=4$. In a similar way, we can compute the vertex $\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(p)$. In order to do this, however, we should take into account that finite contributions are generated in its trace. We do so in the following manner: first, impose the diffeomorphism identity (5.104), which implies that,

$$\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(p)=\Gamma^{\perp}_{g\mathcal{T}}(p)p^\alpha\left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right).$$

Then we compute the unregularized $\Gamma^{\perp}_{g\mathcal{T}}(p)$ by taking the trace of the unregularized $\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(p)$, and only then we renormalize $\Gamma^{\perp}_{g\mathcal{T}}(p)$.

This makes sure that, if finite terms are generated in the trace, $\Gamma^{\perp}_{g\mathcal{T}}(p)$, we will self consistently account for them. We point out that this procedure is necessary to obtain a result that does not break the Ward identities of Weyl symmetry (5.12), which means that the "anomalous trace", generated by finite contributions in the dimensional regularization of $\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}(p)\eta_{\mu\nu}$, needs to be taken into account for renormalization to be consistent with Weyl symmetry, the way we have defined it in this thesis.

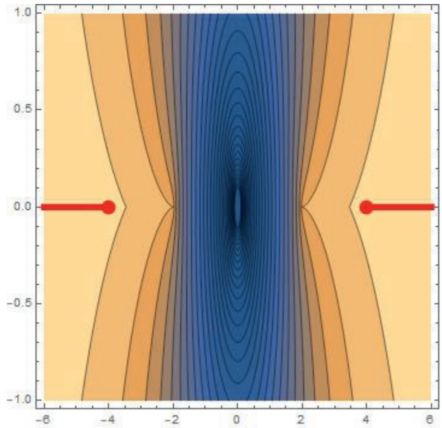


FIGURE 5.3: Analytical structure of the longitudinal part of $\Delta^{\alpha\beta}_{TT}(p)$, as a function of the complex momentum p^2. The colour code denotes the magnitude of the function, black lines are of constant magnitude. In red, beginning at the red dots, are the branch cut discontinuities.

The result of this procedure gives,

$$\Gamma^{\mu\nu\alpha}_{g\mathcal{T},\mathrm{ren}}(p) = \frac{i}{16\pi^2}\frac{(6\xi+1)p^\alpha\left(p^\mu p^\nu - p^2 g^{\mu\nu}\right)}{6p^2}\Bigg(-(6\xi+1)p^2 \times \\ \times \left(\log(4\pi) - \gamma_E + \log\left(\frac{\mu^2}{m^2}\right)\right) \\ + 4\left(m^2 - \xi p^2\right) - \frac{\sqrt{p^2\left(p^2+4m^2\right)}\left(2m^2-(6\xi+1)p^2\right)}{p^2}\times \\ \times \log\left(\frac{\sqrt{p^2\left(p^2+4m^2\right)}+2m^2+p^2}{2m^2}\right)\Bigg). \tag{5.112}$$

5.6 Discussion and conclusion

We want to understand the properties of the 2-point function (5.111). In particular, we split (5.111) in its transverse and longitudinal part, to show the existence of a branch cut discontinuity in the longitudinal part, starting at $-p^2 = 4m^2$ [7].

We compare this with the analogous expression for the 1-loop vacuum polarization of the photon in QED, $\Pi^{\mu\nu}$, which presents a similar pole, which however sits in its transverse part rather than in the longitudinal. The Ward

[7] The form of (5.111) is somewhat misleading, in order to make the branchcut more explicit, one can use the formula, $\arctan(x) = \frac{i}{2}\log\sqrt{\frac{1-ix}{1+ix}}$. In (5.111) we choose to expand the logarythm in the arctan, since this makes explicit that the potentially IR dangerous limit, namely $m \to 0$, is finite.

identities of QED in fact guarantee that no quantum correction is generated in the longitudinal part of the photon propagator. Writing therefore

$$\Pi^{\alpha\beta} = \left(\eta^{\alpha\beta} - \frac{p^\alpha p^\beta}{p^2}\right)\Pi(p^2)\,,$$

we would find that $\Pi(p^2)$ presents similar discontinuities starting at the threshold momentum, $-p^2 = 4m^2$.

In the Weyl gauge theory, however, due to the non compactness of the group, the transversality condition is violated, and the physical excitation only shows up in the longitudinal part. Since there is no transverse source in the scalar theory analyzed in this section, the transverse part of (5.111) only appear to contain a mass renormalization, independent of the external momentum.

The physical meaning of the self-energy is in fact to renormalize the mass, as resumming an infinite series of diagrams leads to the correction to the mass of the gauge field in question (torsion or photon). If the self energy goes to zero for zero external momentum, no mass term is generated, which is indeed the case for QED, while if it goes to a constant it indeed generates a self mass, as can be the case in Higgs symmetry breaking.

We notice that, in the limit in which $m \to 0$, (5.111) becomes purely longitudinal, but the correction to the dilaton mass goes to zero, as can be seen by taking the projection of $\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}(p)$ onto its longitudinal part,

$$\begin{aligned}\Gamma_{\|}(p) = {}& \frac{p_\alpha p_\beta}{p^2}\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}(p) = \\ = {}& \frac{i}{16\pi^2}\frac{p^2}{2}\left((6\xi+1)^2\left(\log(4\pi) - \gamma_E + \log\left(\frac{\mu^2}{m^2}\right)\right) + 4\xi(6\xi+1)\right. \\ & + \frac{2m^2\left(4\xi + 1 - (6\xi+1)\left(\log(4\pi) - \gamma_E + \log\left(\frac{\mu^2}{m^2}\right)\right)\right)}{p^2} \\ & \left. - \frac{(6\xi+1)^2}{p^2}\sqrt{p^2\left(p^2+4m^2\right)}\log\left(\frac{\sqrt{p^2\left(p^2+4m^2\right)} + 2m^2 + p^2}{2m^2}\right)\right),\end{aligned} \tag{5.113}$$

in the limit of zero scalar field mass, $m \to 0$. In the case $m \neq 0$ a constant contribution is generated to $\Delta_{\|}$, which shows that in such a limit the dilaton aquires a mass correction. In both limits, however, a non zero renormalization scale is introduced, but as we have show in the previous section this does not by itself create anomalies in the Ward identities. When the symmetry is explicitely violated, as in the case $m \neq 0$, corrections to the Ward identities are generated, as predicted by (5.80).

Furthermore, when the symmetry is spontaneously broken, the dilaton remain massless, as can be understood from the fact that non p independent term is generated in this limit. This demonstrates that the scale symmetry breaking, if spontaneous does not generate a mass for the dilaton, while when explicit, it does.

Finally, we notice how, in the limit $\xi \to -\frac{1}{6}$, the branch cut discontinuities disappear, as does most of the self energy. Finite contribution might appear in the computation, as setting $\xi = -\frac{1}{6}$ leads to terms proportional to $D-4$. Even such contributions do not violate the Ward identities (5.80), but show how in the conformally coupled case no physical dilaton is present.

As a final exercise, we perform the analysis done for $\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}$ for $\Gamma^{\mu\nu\beta}_{g\mathcal{T}}$, to arrive to similar conclusions. In fact, the tranverse part of $\Gamma^{\mu\nu\beta}_{g\mathcal{T}}$, as can be easily seen from Eq. (5.112), gives,

$$\left(\eta_{\alpha\beta} - \frac{p_\alpha p_\beta}{p^2}\right)\Gamma^{\mu\nu\alpha}_{g\mathcal{T}} = 0\,. \tag{5.114}$$

The physical branchcut therefore sits in its longitudinal component, and has, thanks to the Ward-Takahashi identities (5.80), the same properties.

5.7 Conclusion

We provided evidence suggesting that the longitudinal part of the compensating field of Weyl transformations, $\mathcal{T}_\mu$, should be regarded as the dilaton field whose existence is implied by the Weyl anomaly [24, 25, 114]. In fact, adding the gauge compensating field leads to a renormalized theory where the Ward identities holds, even though anomalous contributions emerge both in the dilatation current, $\langle D^\mu \rangle$, and in the energy-momentum tensor, $\langle T^{\mu\nu} \rangle$.

To support this claim, we have found that the longitudinal component of the self energy $\mathcal{T}_\mu$ acquires a branchcut discontinuity which extends to the infrared if scale symmetry is spontaneously broken. This supports our discussion, in section 4, which claims the existence of a massless degree of freedom, coupling to the Weyl anomaly, if the symmetry is broken spontaneously, *e.g.* by the mechanism of chapter 3.

In this computation, anomalous terms are still generated by the loop expansion of Feynman diagrams. These contributions generate the anomalous trace of the energy-momentum tensor, which should be accounted for in the renormalized theory, as we discuss in the appendix A. We found that, in order to respect the Ward-Takahashi identities for Weyl symmetry, we actually need to include this contribution, since the same sort of terms are generated in the diagrammatic expansion of the dilatation current. We conjecture that this behaviour is actually generic, and holds beyond the computation presented here.

Finally, we attribute the anomalous terms that one finds if the torsion field is not included, as signaling the existence of a Goldstone degree of freedom, which represents the Goldstone mode of broken scale transformations. The anomaly in the trace of the energy momentum tensor, therefore, represents the breaking of a global symmetry, *i.e.* scale symmetry, but is consistent with the gauge symmetry defined by (1.4), since the relevant Ward-Takahashi identities seem to hold.

These conclusions are still preliminary, as more evidence in support of them is needed. First, one must compute the 3-points functions,

$$\Gamma^{\alpha\beta\gamma}_{\mathcal{T}\mathcal{T}\mathcal{T}}\,,\Gamma^{\mu\nu\beta\gamma}_{g\mathcal{T}\mathcal{T}}\,,\Gamma^{\mu\nu\lambda\sigma\gamma}_{gg\mathcal{T}}\,,\Gamma^{\mu\nu\lambda\sigma\tau\rho}_{ggg}\,,$$

defined in an analogous way as (5.82) and show that still the Ward-Takahashi identities (5.12) hold in a renormalized perturbation theory. This calculation should prove that no anomalous terms quadratic in the metric and torsion fields is generated on the right hand side of (5.71), even though these are generated in the energy momentum trace.

The second step to take is to study the Ward identities (5.80) in the case of an interacting theory, such as QED or introducing the term $\lambda\phi^4$ in the action (5.73). This would produce a modification in the Ward identities (5.80) which in dimensional regularization would yield,

$$\nabla_\mu\langle D^\mu\rangle + \langle T^\mu_\mu\rangle = \frac{m^2}{2}\langle\phi^2\rangle + (D-4)\lambda\langle\phi^4\rangle\,. \tag{5.115}$$

Similarly, in the case of QED, we find anomalous term contributing to the dilatation current,

$$\begin{aligned}\nabla_\mu\langle D^\mu\rangle+\langle T^\mu_\mu\rangle &= m\langle\bar\psi\psi\rangle + (D-4)e\langle A_\mu\lambda\bar\psi\gamma^\mu\psi\rangle\,,\\ D^\mu &= (D-4)A^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\,,\end{aligned} \tag{5.116}$$

Understanding the role of the terms proprtional to $D-4$ in (5.115–5.116), and the possible finite contribution they generate in dimensional regularization is a non trivial task. How this finite contributions modify the spectrum and theory for the dilaton, and if and how identities (5.115–5.116) can be used to gather information about the perturbative expansion is essential to understand the anomaly induced by the running of the coupling constants in a quantum field theory.

This last chapter provides the first step in these directions, and provides enough evidence for us to conjecture that the identities (5.115–5.116) actually hold, as was the case for the non interacting theory.

Appendix A: Derivation of the $g\mathcal{T}$ proper vertex and the loop integrals

Here we discuss how to perform the loop integrals that lead to the equations (5.112–5.111) in the main text. We will review how to reduce the tensorial integrals (5.101–5.102) to the scalar loop integral (5.99), and compute explicitely the local integrals $A^{\mu\cdots}(p)$.

Let us begin by considering the loop integrals,

$$
\begin{aligned}
B_0(p) =& \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\left(l^2 + m^2 - 2x\, p \cdot l + x\, p^2\right)^2} = \\
=& \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^D l'}{(2\pi)^D} \frac{1}{\left(l'^2 + m^2 - x(x-1)\, p^2\right)^2} = \\
=& \frac{i}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)} \int_0^1 \mathrm{d}x \, \left(-m^2 + x(x-1)\, p^2\right)^{D/2-2} = \\
= -& \frac{i}{16\pi^2} \left[\frac{2\mu^{D-4}}{D-4} + \gamma_E - \log\left(\frac{4\pi\mu^2}{m^2}\right) - 2 + \right. \\
+& \left. \frac{\sqrt{p^2\left(p^2 + 4m^2\right)}}{p^2} \log\left(\frac{\sqrt{p^2\left(p^2+4m^2\right)} + 2m^2 + p^2}{2m^2} \right) \right].
\end{aligned}
\tag{5.117}
$$

Similarly we have,

$$
\begin{aligned}
A(p) &= \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{1}{\left((l-p)^2 + m^2\right)} = && (5.118) \\
&= \frac{i}{(4\pi)^2} \left(\frac{2\mu^{D-4}}{D-4} + \gamma_E - 1 + \log\left(\frac{m^2}{4\pi\mu^2}\right)\right) m^2 \,, \\
A^\mu(p) &= \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu}{\left((l-p)^2 + m^2\right)} && (5.119) \\
&= \frac{i}{(4\pi)^2} \left(\frac{2\mu^{D-4}}{D-4} + \gamma_E - 1 + \log\left(\frac{m^2}{4\pi\mu^2}\right)\right) m^2 p^\mu \,, \\
A^{\mu\nu}(p) &= \int \frac{\mathrm{d}^D l}{(2\pi)^D} \frac{l^\mu\, l^\nu}{\left((l-p)^2 + m^2\right)} = && (5.120) \\
&= \frac{i}{(4\pi)^2} \left(\frac{2\mu^{D-4}}{D-4} + \gamma_E - 1 + \log\left(\frac{m^2}{4\pi\mu^2}\right)\right) m^2 p^\mu p^\nu + \\
&- \frac{1}{4} \frac{i}{(4\pi)^2} \left(\frac{2\mu^{D-4}}{D-4} + \gamma_E - \frac{3}{2} + \log\left(\frac{m^2}{4\pi\mu^2}\right)\right) m^4 g^{\mu\nu} \,.
\end{aligned}
$$

This is then to be combined with the reduction formulas, for the tensorial integrals $B^{\mu_1 \cdots \mu_n}$, defined in (5.100–5.102), to obtain the unregularized expressions for $\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}, \Gamma^{\mu\nu\alpha}_{g\mathcal{T}}$.

First, we can easily compute $\Gamma^{\alpha\beta}_{\mathcal{T}\mathcal{T}}$, by noticing that,

$$
B^\mu(p) = X(p) p^\mu = \frac{p^\mu}{2p^2} \left(p^2 B_0(p) + A(p) - A(0) \right) \,,
$$

as can be verified by analysing the projection, $p_\mu B^\mu(p)$, and applying the formula $p \cdot l = \frac{1}{2}\left(p^2 + l^2 - 2(p-l)^2\right)$ once. Then the unknown quantity $X(p)$ will yield the right hand side.

Slightly more involved is the computation of the vertex $\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}$. This is done by repetedly applying the reduction formulas as before, writing, for the tensorial integrals in (5.100–5.102),

$$\begin{aligned} B^{\mu\nu}(p) =& X_2(p) g^{\mu\nu} + Y_2(p) p^\mu p^\nu \,, \\ B^{\mu\nu\lambda}(p) =& X_3(p) g^{(\mu\nu} p^{\lambda)} + Y_3(p) p^\mu p^\nu p^\lambda \,, \end{aligned} \tag{5.121}$$

where we took into account that both these tensors are totally symmetric in their indices. Projecting these relations and simplifying the loop integrals by means of $p \cdot l = \frac{1}{2}\left(p^2 + l^2 - 2(p-l)^2\right)$, we can solve for the unknown coefficients X_2, X_3, Y_2, Y_3 in (5.121). This is clearly possible since the right hand side of (5.121) is a linear combination of linearly independent tensors.

The solution,

$$X_2 = \frac{1}{(D-1)}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) B^{\mu\nu}(p)\,, \tag{5.122}$$

$$Y_2 = \frac{1}{(D-1)p^2}\left(D\frac{p_\mu p_\nu}{p^2} - \eta_{\mu\nu}\right) B^{\mu\nu}(p)\,, \tag{5.123}$$

$$X_3 = \frac{3}{(D-1)p^2}\left(\eta_{(\mu\nu}p_{\lambda)} - \frac{p_\mu p_\nu p_\lambda}{p^2}\right) B^{\mu\nu}(p)\,, \tag{5.124}$$

$$Y_3 = \frac{1}{(D-1)p^4}\left((D+2)\frac{p_\mu p_\nu p_\lambda}{p^2} - 3\eta_{(\mu\nu}p_{\lambda)}\right) B^{\mu\nu\lambda}(p)\,, \tag{5.125}$$

which can be further reduced by repetitively using the reduction formulas,

$$p_\mu B^{\mu\nu}(p) = \frac{1}{2}\left(p^2 B^\nu(p) + A^\nu(p) - A^\nu(0)\right)\,, \tag{5.126}$$

$$p_\mu B^{\mu\nu\lambda}(p) = \frac{1}{2}\left(p^2 B^{\nu\lambda}(p) + A^{\nu\lambda}(p) - A^{\nu\lambda}(0)\right)\,, \tag{5.127}$$

$$\eta_{\mu\nu} B^{\mu\nu}(p) = \left(A(p) - m^2 B(p)\right)\,, \tag{5.128}$$

$$\eta_{\mu\nu} B^{\mu\nu\lambda}(p) = \left(A^\lambda(p) - m^2 B^\lambda(p)\right)\,. \tag{5.129}$$

This procedure allows to calculate the unregularized vertex $\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}$, however, before subtracting the divergence, we need to take into account the trace anomaly, namely any possible finite contribution in the dimensional expansion of the vertex $\eta_{\mu\nu}\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}$. Indeed, the property we would like to preserve is, clearly,

$$\Gamma^{\mu\nu\alpha}_{g\mathcal{T},\mathrm{ren}}\eta_{\mu\nu} = \left[\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}\eta_{\mu\nu}\right]_{\mathrm{ren}}\,,$$

which simply means that the finite terms generated in the loop expansion of the trace should be included in the renormalized stress energy tensor.

To illustrate this procedure, consider the unregularized $\Gamma^{\mu\nu\alpha}_{g\mathcal{T}}$, it turns out that its form is,

$$\Gamma^{\mu\nu\alpha}_{g\mathcal{T},\mathrm{div}} = p^{\alpha}\left(\eta^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2}\right)\left(\frac{\mu^{D-4}}{D-4} + \textit{finite}\right), \tag{5.130}$$

where *finite* denotes the finite contribution to the vertex. Then, the trace will be,

$$\Gamma^{\mu\nu\alpha}_{g\mathcal{T},\mathrm{div}}\eta_{\mu\nu} = p^{\alpha}\,(D-1)\left(\frac{\mu^{D-4}}{D-4} + \textit{finite}\right) = p^{\alpha}\left(\frac{3\mu^{D-4}}{D-4} + 1 + 3\times\textit{finite}\right), \tag{5.131}$$

where the extra $+1$ on the right hand side of (5.131) comes from the expansion of $\frac{D-1}{D-4}$ around $D \simeq 4$. This last step simply insures that we are including *all* finite terms in the expansion near $D = 4$. Note that this is precisely the way the trace anomaly is calculated for example in [17].

In the case at hand, since Eq. (5.131) shows that $\Gamma^{\mu\nu\alpha}_{g\mathcal{T},\mathrm{div}}(p)$ is transverse in the indices μ, ν, we can compute its renormalized form by computing its trace and keeping only the finite part of it. This procedure leads to Eq. (5.112).

Derivation of the gg proper vertex and verification of the remaining Ward-Takahashi identity

In this section we are going to derive and write down the Ward-Takahashi identity that is associated with the 2-point vertex

$$\Gamma^{\mu\nu}_{gg\lambda\sigma}(x,y) \equiv {}^{\mu\nu}\Pi_{\lambda\sigma}(x,y) = \frac{\delta\langle T^{\mu\nu}(x)\rangle}{\delta g^{\lambda\sigma}(y)}. \tag{5.132}$$

We limit ourselves to the case of massless field ϕ, since it is already instructive, and the computation at finite m is significantly more involved. The advantage is that no local contributions are generated which makes the computation in the massless limit somewhat easier.

The unregularized expression for $\Gamma^{\mu\nu}_{gg\,\lambda\sigma}(x,y)$ in momentum space is,

$$\begin{aligned}
\Gamma^{\mu\nu\lambda\sigma}_{gg} \;=\; & \frac{i}{16\pi^2}\frac{1}{1800}\times \\
& \times\Bigg\{ 2p^4\eta^{\lambda\sigma}\eta^{\mu\nu}\Bigg[15\Big(40\xi(3\xi+1)+3\Big)\left(\frac{1}{\epsilon}+\log\left(\frac{\mu^2}{p^2}\right)\right) \\
& \qquad +8\Big(25\xi(18\xi+5)+6\Big)\Bigg] \\
& +p^4\left(\eta^{\lambda\nu}\eta^{\mu\sigma}+\eta^{\lambda\mu}\eta^{\nu\sigma}\right)\left[\frac{15}{\epsilon}+15\log\left(\frac{\mu^2}{p^2}\right)+46\right] \\
& +1800p^\lambda p^\mu p^\nu p^\sigma\Bigg[\frac{1}{15}\Big(10\xi(3\xi+1)+1\Big)\left(\frac{1}{\epsilon}+\log\left(\frac{\mu^2}{p^2}\right)\right) \\
& \qquad +4\xi^2+\frac{10\xi}{9}+\frac{47}{450}\Bigg] \\
& -2p^2\left(p^\mu p^\nu\eta^{\lambda\sigma}+p^\lambda p^\sigma\eta^{\mu\nu}\right)\Bigg[15\Big(40\xi(3\xi+1)-3\Big)\left(\frac{1}{\epsilon}+\log\left(\frac{\mu^2}{p^2}\right)\right) \\
& \qquad +8\Big(25\xi(18\xi+5)+6\Big)\Bigg] \\
& -p^2\left(p^\nu p^\sigma\eta^{\lambda\mu}+p^\mu p^\sigma\eta^{\lambda\nu}\right)\left(\frac{15}{\epsilon}+15\log\left(\frac{\mu^2}{p^2}\right)+46\right) \\
& -p^2\left(p^\lambda p^\nu+p^\lambda p^\mu\eta^{\nu\sigma}\right)\eta^{\mu\sigma}\left(\frac{15}{\epsilon}+15\log\left(\frac{\mu^2}{p^2}\right)+46\right)\Bigg\}\,,
\end{aligned} \tag{5.133}$$

where we defined the parameter,

$$\frac{1}{\epsilon} = -\frac{2\left(e^{-\gamma_E}\mu^2\right)^{(D-4)/2}}{D-4}\,. \tag{5.134}$$

The expression (5.133) satisfied the Ward identities,

$$p_\mu\Gamma^{\mu\nu\lambda\rho}_{gg} = p_\nu\Gamma^{\mu\nu\lambda\rho}_{gg} = p_\lambda\Gamma^{\mu\nu\lambda\rho}_{gg} = p_\sigma\Gamma^{\mu\nu\lambda\rho}_{gg} = 0\,, \tag{5.135}$$

$$ip_\alpha\Gamma^{\mu\nu\;\alpha}_{g\mathcal{T}} + \eta_{\lambda\sigma}\Gamma^{\mu\nu\lambda\rho}_{gg} = 0\,, \tag{5.136}$$

again if the finite traces generated in the expansion are taken into account.

In order to renormalize the vertex function, we notice that the identity (5.135), together with the bose symmetry $\Gamma_{gg}^{\mu\nu\lambda\rho}(p) = \Gamma_{gg}^{\lambda\rho\mu\nu}(-p)$ implies the following expansion for the graviton self energy,

$$\begin{aligned}\Gamma_{gg}^{\mu\nu\lambda\rho} &= A\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(\eta^{\lambda\sigma} - \frac{p^\lambda p^\sigma}{p^2}\right) \\ &\quad + B\left(\eta^{\mu(\lambda} - \frac{p^\mu p^{(\lambda}}{p^2}\right)\left(\eta^{\sigma)\nu} - \frac{p^{\sigma)} p^\nu}{p^2}\right)\end{aligned} \tag{5.137}$$

which allows us to extract the finite terms of the coefficients A and B, in dimensional regularization, in the following manner: first compute the finite trace of $\Gamma_{gg}^{\mu\nu\lambda\rho}$, then require the following conditions for the renormalized coefficients $A_{\rm ren}$, $B_{\rm ren}$,

$$\left[\eta_{\mu\nu}\Gamma_{gg}^{\mu\nu\lambda\sigma}\right]_{\rm ren} = (3\,A_{\rm ren} + B_{\rm ren})\left(\eta^{\lambda\sigma} - \frac{p^\lambda p^\sigma}{p^2}\right), \tag{5.138}$$

$$\left[\eta_{\mu\lambda}\Gamma_{gg}^{\mu\nu\lambda\sigma}\right]_{\rm ren} = (A_{\rm ren} + 2B_{\rm ren})\left(\eta^{\nu\sigma} - \frac{p^\nu p^\sigma}{p^2}\right). \tag{5.139}$$

The relations (5.138–5.139) guarantee that any finite trace that is generated in the loop expansion is accounted for, in a manner that respects both the diffeomorphisms and the Weyl symmetry Ward-Takahashi identities. This procedure, as before, makes sure that we account for the anomalous trace of the energy-momentum tensor, and we must do so in order to respect the Weyl symmetry Ward Takahashi identities.

This leads to the expressions, for the coefficients $A_{\rm ren}$, $B_{\rm ren}$,

$$\begin{aligned}A_{\rm ren} &= \frac{i}{16\pi^2}\Bigg[-\frac{p^4}{60}\left(120\xi^2 + 40\xi + 3\right)\left(\log\left(\frac{p^2}{4\pi\mu^2}\right) + \gamma_E\right) \\ &\quad + \frac{p^4}{60}\left(64\xi^2 + 8\xi - 1\right)\Bigg]\end{aligned} \tag{5.140}$$

$$\begin{aligned}B_{\rm ren} &= \frac{i}{16\pi^2}\Bigg[-\frac{p^4}{60}\left(\log\left(\frac{p^2}{4\pi\mu^2}\right) + \gamma_E\right) \\ &\quad + \frac{p^4}{60}\left(48\xi^2 + 16\xi + 3\right)\Bigg].\end{aligned} \tag{5.141}$$

We point out once more that the procedure to extract the finite trace carried out in this section, does not affect the terms proportional to $\log\left(\frac{\mu^2}{p^2}\right)$, but only the finite terms generated in the expansions.

Chapter 6

Summary

In this thesis we presented a systematic study of a generic theory of gravity, endowed with additional scalar and vectorial degrees of freedom, exhibiting a local version of scale invariance, which we refer to as Weyl symmetry.

The reasons why Weyl symmetry is a desirable feature in constructing models of nature at high energy, are multiple. As we saw, in a Weyl invariant theory, a special place is reserved to dimensional regularization, which appears to respect the symmetry and only introduces logarithmic running of the coupling constants of the theory. This then allows for large hierarchies to develop between different scales [11] and coupling constants, and can be used to solve the gauge hierarchy problem.

Secondly it is well known now that the Higgs mass parameter sits at the edge of the stability bound [115, 116], and its quartic coupling seems to run to zero, which is its conformal value, near the Planck scale. This suggests that the symmetry is restored near the Planck scale, or depending on the renormalization group flow at a intermediate scale between the electroweak symmetry breaking scale and the Planck scale. Addition of extra scalar or vectorial degrees of freedom can change the stability of the Higgs potential, by modifying the running of the theory with scale. It has been found that conformal extensions of the standard model of this kind are capable of solving the stability issue of the standard model [117], whilst remaining natural.

Also note that, if Weyl symmetry is realised at the quantum level, then the expectation value of the cosmological constant, $\langle \Lambda \rangle = 0$. Therefore, a non vanishing cosmological constants signals a breaking of the symmetry, which might render the cosmological constant technically natural [74].

All these theoretical argument point to the hypothesis that Weyl symmetry is realised in nature, and open to the possibility of testing Weyl symmetry experimentally. Since it might imply new physics just above the electroweak symmetry breaking (as for example in [117]). Furthermore, as we have argued in this thesis, a Weyl invariant theory predicts additional gravitational degrees of freedom that might become accessible to gravitational waves observatories such as LISA. Finally, the symmetry is somehow special, since it only allows operators of dimension 4 to be written in the Lagrangian, thus stronlgy limiting the number of coupling constants that are needed to define it. The principal freedom in specifying the theory is the choice of the degrees of freedom it contains.

We found that the minimal field content to achieve a Weyl invariant theory is a metric field $g_{\mu\nu}$, a scalar field ϕ, and a vector field $\mathcal{T}_\alpha$ [72]. There exist two realizations of this theory, differing in the geometrical interpretation of the vector field $\mathcal{T}_\mu$. In the old view of Weyl [118], this vector is interpreted as representing non metricity,

$$\nabla_\mu g_{\alpha\beta} = 2\mathcal{T}_\mu g_{\alpha\beta}\,,$$

while in our view it has the geometrical meaning of torsion trace,

$$\Gamma^\lambda{}_{[\mu\nu]} = \delta^\lambda_{[\mu}\mathcal{T}_{\nu]}\,.$$

Whatever the geometrical interpretation, the gauge field has the physical meaning of defining, for a point-like observer moving on the curve $\gamma(\lambda)$, the parallel transport properties of dimensionfull quantities, via the relation,

$$\dot\gamma^\mu\overline{\nabla}_\mu M(x) = \dot\gamma^\mu\partial_\mu M(x) + \Delta_M\dot\gamma^\mu\mathcal{T}_\mu M(x)\,,\dot\gamma^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\,, \tag{6.1}$$

and thus defines a geometrical notion of scale. This leads to a well defined geometric theory, that extends the concepts in general relativity in such a way that no reference to a fundamental length or energy scale is ever made [72]. The requirement of metric compatibility singles out the torsion field as the unique possibility realising this idea.

In Chapter 2 we describe the geometry that emerges from these assumptions, and how it modifies certain properties of general relativity, namely the study of Jacobi fields, to accomodate for the gauge symmetry. In the end of the section, we show that the conformal group admits large gauge transformations, which can and will modify the Gauss-Bonnet topological charge (2.91), thus leading to topological effects that are physical.

It is quite obvious that we do not live in a scale invariant universe, but a valid assumption is that scale symmetry gets broken spontaneously and is restored if we observe the universe at very short scales. The most natural way this could happen is via the Coleman-Weinberg mechanism [1] according to which quantum corrections can spontaneously generate a symmetry breaking minimum in the effective potential. This view is in accordance with the asymptotic safety idea, which aims to define quantum gravity non perturbatively, as a conformal field theory living at the fixed point of the renormalization group flow [14, 15]. Near the fixed point, fields condense and generate a primordial notion of scale, which is the assumption we follow in chapter 3, where we show that, if a non vanishing mass or energy scale is generated, either in the space-time curvature, $\langle R\rangle$, or in the field theory, $\langle\phi^2\rangle$, then the system gravity-scalar can describe the phenomenon of cosmic inflation, with good accordance with the data and with desirable features [91]. In this model, the theory contains 3 scalar fields, the physical inflaton, ϕ, the gravitational scalaron R and the goldstone mode of broken dilatations. One such scalar field is a gauge mode, therefore unphysical, however the others are physical. Amongst these, there exist a flat direction, which is a mixture of the scalar field ϕ and the longitudinal torsion $\mathcal{T}^L_\mu = \partial_\mu\theta$, singling out uniquely the inflaton as the non flat direction.

The Weyl symmetry in this model is linked to a hyperbolic geometry of field space, which suggests it might have universal features, similar to the ones studied in [28, 29]. In fact, the model proposed in [91] has a similar geometry to these, poossessing negative curvature and Euclidean signature, which still awaits a deeper understanding.

In chapter 4 we demonstrate that, if space-time torsion exists and is physical, it can be measured by gravitational wave instruments. In particular, since in this theory there exists a physical flat direction, it is possible that scalar massless modes produced in extreme astrophysical events propagate to Earth.

The scalar massless particle evades the constrains that are usually enforced on these kinds of models [96]. This is because it does not couple to fermionic or gauge standard model fields other than through the trace anomaly. In addition, as we discuss in section 4.6.1 its coupling to the Higgs field is suppressed by the Planck scale, $\propto \frac{k^2}{M_P^2}$, which renders it virtually undetectable at the LHC. However, if gravitational or matter energy densities of $\mathcal{O}(M_P)$ are generated, this particle might get produced, and propagate at large distances. This would render it detectable once the next generation of space-based gravitational waves observatories come online.

Finally, in chapter 5, we analyse some of the quantum properties of the theory. We found further evidence supporting the claim that a physical, massless mode exists in the spectrum of the theory, which is also suggested by appearence of the Weyl anomaly [24, 25, 114]. It might acquire a mass if the symmetry is broken explicitely, *i.e.* if it is broken at the classical level for example by a mass term, but does not if it is broken by quantum effects (when the renormalization scale μ is introduced).

We studied the Ward identities obeyed by the 2-points proper vertices renormalizing the self energy of the gauge field $\mathcal{T}_\mu$ and the metric field $g_{\mu\nu}$ and found that they are satisfied even when finite contributions are generated in the loop expansion. The physical meaning of this is that there always exists a linear combination of the scalar fields contained in $\mathcal{T}_\mu^L = \partial_\mu \theta$ and the scalar graviton h_μ^μ that remains not excited, while other become physical and interact with fundamental matter scalars such as the Higgs. This property can be traced back to the gauge symmetry, and we conjecture should hold non perturbatively, at least when the renormalization group flow reaches a critical point and conformal symmetry is restored by quantum fluctuations.

This thesis provides evidence that non trivial physical relations hold between the various scalar fields, both matter and gravitational, contained in the theory. A particular role is played by the way such scalars mix with the gravitational fields, mixing that appears at the classical level, according to our discussion in chapter 2, and is modified by quantum effects, as chapter 5 demonstrates. Another important observable is the Weyl charge defined in (2.98–2.100) which is related somewhat to the squeezing of the quantum state of the theory. We have demonstrated how these observables have a non trivial relation with Weyl symmetry, which we are just beginning to understand and certainly deserves more investigation.

Chapter 7

English Summary

Symmetries are of crucial importance in physics, for the reason that they allow to simplify complex problems, and gather exact information about the dynamical behaviour of the theory. A classic example is rotational invariance of the action, which implies the conservation of angular momentum, or time translational invariance that implies the conservation of energy. This concept is founded on the mathematical theory developed by Emmy Noether, who showed that when a physical system exhibit continuous symmetries, there always exists a corresponding conserved quantity.

Modern physics has not abandoned this concept, but rather refined it. Indeed, the construction of the modern theories of particle physics, the Standard Model and General Relativity, are based on two symmetry principles: that of Lorentz invariance and gauge symmetry. The first, upon which the theory of Special Relativity is based, is an extension of rotational and translational invariance to a *space-time*, to incorporate time translations and *boosts*. The boost transformations relate different observers moving with constant speed, while the rest of the Lorentz group is the set of geometrical translations and rotations.

Lorentz symmetry, as the symmetries treated in Noether's theory, is a global symmetry. For example, a sphere exhibits global rotational symmetry because it is the same no matter what direction one looks at it, which is a property specific of that system. On the other hand, the observables of a theory should be the same no matter the direction one is looking from, which means that the description of physics should be invariant under arbitrary rotations. This is an example of what is called a gauge (local) symmetry: in order to make sure our description (not the state of the physical system) is invariant under rotations, we have to introduce a redundancy. In most cases, such a redundancy consists of an arbitrary function that specifies some geometrical coordinate, for example the angle we are observing the system at. Since the physical properties should not depend on the particular choice of this free function, the action should be the same whatever choice is made. This would reflect our freedom of choosing the direction we look at the system, but should not impact the physical properties we measure.

When Lorentz symmetry is combined with gauge symmetry, we obtain the theory of General Relativity. The generalised Lorentz transformations, which go by the name of diffeomorphisms, are then used to relate different observers, not necessarily moving at constant speed. So for example they relate observers accelerating with respect to one another. This strategy can be extended to the Standard Model's fields, however, such description is not fully understood from a theoretical perspective, as many open problems (renormalizability, unitarity...) are yet unsolved.

In this thesis I explore the possibility of extending the Lorentz symmetry to the conformal group. The conformal group is an extension of the Lorentz group, to all transformations that preserve angles, and therefore contains arbitrary rescaling of lengths. It includes generalised scale transformations, which might depend on the space-time coordinates. There are multiple reason for considering this type of generalised symmetry, most notably the fact that the Standard Model nearly respect the symmetry, at the classical level, and seems to flow towards a conformally invariant theory at high energies. This would then suggest that scale invariance is restored at short distances, and lost by a spontaneous symmetry breaking mechanism, at some intermediate scale between the electro weak scale and the Planck scale.

The underlying assumption that such an hypothesis implies is that there exists no fundamental length scale in nature, but all the dimensionful quantities that we observe can only be defined in relation to one another. Because of this, no intrinsic scale should exist in this theory, and the symmetry has to be broken by the spontaneous creation of at least two condensates (the symmetry breaking parameter is then the ratio of this two scales). I study a simple realisation of this mechanism in Chapter 3, showing it can quite generically lead to a model for cosmic inflation that is in good agreement with the data and has desirable features.

In this model, the first scales that are generated are a gravitational energy density and the condensate of the inflaton field, a scalar particle, perhaps created by quantum fluctuations around a flat space-time. Such state has a lot of gravitational energy and is unstable, which means the energy eventually gets transferred to the inflaton field, and ultimately to Standard Model particles via the reheating process. This process is generic since scale symmetry imposes strong constrains on the kind of operators one can include in the theory, to the extent that the only freedom of choice in the theory is in the number and character of degrees of freedom one wants to include. This renders the theory appealing, as its predictions are controlled only by a handfull of parameters (3 in the minimal case).

An important feature of any spontaneous symmetry breaking pattern is the existence of Goldstone modes, which are massless particles that appear in the low energy spectrum if a spontaneous symmetry breaking happens at high energies. In other words, Goldstone modes are excitations along the symmetry direction, which is why they remain massless, and survive in the low energy effective theory. When scale symmetry is broken spontaneously, one can rigorously show that a massless scalar field is generated, similar to the pions that stem from the chiral symmetry breaking in QCD, but neutral under all Standard Model charges. Because of the symmetry, this scalar particle has to couple universally to all matter, and can be observed through techniques available today, most notably by gravitational waves observatories, as I discuss in Chapter 4. Even though it is massless, however, its weak coupling to Standard Model particles hides it from all particle's physics experiments performed so far, and since it interacts at tree level only with scalar fields, it will not partake in any process involving fermions or gauge fields. However, extreme conditions such the ones existing near a black hole or a neutron star will lead to the production of this mode, which would render it accessible by space based gravitational wave observatories, such as LISA.

Furthermore, as I discuss in details in Chapter 5, if this particle should be indeed observed to be massless, it would signify that the scale symmetry is broken spontaneously, while if it would be observed to be massive, it would signify the existence of fundamental length scales in the theory, which are explicit sources of scale symetry breaking. I illustrate this by showing that a mass gap in the self energy of the Goldstone modes is generated by explicit symmetry breaking parameters, in the example of Chapter 5 a constant mass parameter. Such a information would be an unprecedented constrain on the physics in place at the shortest scales, which can be perhaps performed in a model independent way, through the effective field theory approach discussed in Chapter 4.

Finally, thoughout this thesis I provide evidence that non trivial physical relations hold between the various scalar fields, both matter and gravitational, contained in the theory. A particular role is played by the way such scalars mix with the gravitational fields, mixing that appears at the classical level, according to my discussion in Chapter 2, and is modified by quantum effects, as Chapter 5 demonstrates. I have demonstrated how these observables have a non trivial relation with conformal symmetry, which we are just beginning to understand and certainly deserves more investigation.

Chapter 8

Nederlandse samenvatting

Symmetrieën zijn zeer belangrijk in de natuurkunde, aangezien ze complexe problemen sterk kunnen vereenvoudigen. Daarbij kunnen ze exacte informatie vergaren over het dynamische gedrag van de onderliggende theorie. Een typisch voorbeeld hiervan is de rotatie invariantie van de actie, die de wet van behoud van impulsmoment impliceert, of tijd translatie invariantie, die de wet van behoud van energie impliceert. Dit concept heeft als fundament de wiskundige theorie ontwikkeld door Emmy Noether. Zij liet zien dat een fysisch systeem met een continue symmetrie, een corresponderende behouden grootheid heeft.

De moderne natuurkunde heeft dit concept uitgediept en verder ontwikkeld. Onder andere de moderne theorie van deeltjesfysica, het Standaard Model, en de Algemene Relativiteitstheorie zijn gebaseerd op twee symmetrieën: Lorentz- en Ijksymmetrie. De eerste, die de basis is van de Speciale Relativiteitstheorie, is een uitbreiding van de rotatie en translatie invariantie in de ruimtetijd. Dit introduceert tijd translaties en *boosts*. De boosts relateren verschillende waarnemers die zich met een constante snelheid voortbewegen. De rest van de Lorentz transformaties zijn de geometrische rotaties en translaties.

De Lorentz symmetrie is een globale symmetrie. Bijvoorbeeld, een sfeer heeft globale rotatie symmetrie omdat, onafhankelijk van de waarnemingshoek, het hetzelfde blijkt te zijn. Dit is een specifiek eigenschap van het systeem. Daar staat tegenover dat de observabelen van een fysisch systeem horen hetzelfde te zijn, onafhankelijk van de waarnemingshoek. Dit is wat we een lokale- of Ijksymmetrie noemen. Om een beschrijving van de natuur (dus niet de staat van het fysisch systeem zelf) te hebben die invariant is onder rotaties moeten wij een overtolligheid introduceren. Meestal is deze overtolligheid een arbitraire functie die een geometrische coordinaat specificeert, bijvoorbeeld de waarnemingshoek. De actie moet dus onveranderd blijven onder de keuze van de vrije functie aangezien de fysische eigenschappen van het systeem onafhankelijk zijn van deze keuze. Elke waarneming van de eigenschappen zal dus hetzelfde resultaat geven, onafhankelijk van de waarnemingshoek.

De combinatie van de Lorentz- en Ijksymmetrie wordt beschreven door de Algemene Relativiteitstheorie. De algemene Lorentz transformaties, ofwel diffeomorfismes, relateren nu verschillende waarnemers die versnellingen kunnen ondervinden. Deze aanpak kan gehanteerd worden om veldtheorieën op te bouwen, al is dit niet volledig begrepen aangezien er een aantal theoretische problemen onopgelost blijven(renormalizatie, unitariteit, ...).

In dit proefschrift onderzoek ik de hoekgetrouwe groep. De hoekgetrouwe groep is een uitbreiding van de Lorentz groep die transformaties toevoegt die hoeken onaangetast laten. Als gevolg hiervan bevat deze groep arbitraire herschalingen van lengtes. Dit betekent dat algemene schaal-transformaties, die afhankelijk van ruimtetijd coordinaten kunnen zijn, onderdeel zijn van deze groep. Er zijn meerdere redenen om deze veralgemeniseerde symmetrie te bestuderen. De meest belangrijke reden is het feit dat het Standaard Model, op klassieke niveau, bijna aan deze symmetrie voldoet en, bij hoge energieën, naar een hoekgetrouw invariante theorie lijkt te vloeien . Dit suggereert dat schaal invariante geldig is op zeer kleine lengteschalen en vervolgens, door middel van een spontane symmetriebreking mechanisme, verloren gaat op een lengteschaal tussen de Elektrozwakke schaal en de Planck schaal.

De onderliggende aanname van deze hypothese is dat er geen fundamentele schaal bestaat in de natuur. Dit betekent dat alle waargenomen dimensievolle grootheden alleen beschreven kunnen worden in relatie tot mekaar. Dit houdt in dat er geen intrinsieke lengteschaal bestaat voor deze theorie en de symmetrie spontaan gebroken moet worden door de creatie van tenminste twee condensaten (de symmetriebreking parameter is dan de ratio van deze twee schalen). In Hoofdstuk 2 bestudeer ik een simpel voorbeeld van dit mechinisme en laat ik zien dat dit, in een algemene wijze, kan leiden tot een model voor kosmische inflatie. Dit simpel model komt goed overeen met kosmologische waarnemingen en heeft een aantal gewenste eigenschappen.

In dit model worden twee schalen gegenereerd, de energiedichtheid van de zwaartekracht en het condensaat van het inflaton veld, een scalair veld, mogelijk gecreëerd door kwantum fluctuaties rond een vlakke ruimtetijd. Deze fysische staat heeft een hoge zwaartekracht energie en is instabiel. Dit betekent dat een deel van deze energie naar het inflaton veld wordt overgedragen om vervolgens bij het Standaard Model terecht te komen door middel van het reheating process. Dit process is algemeen aangezien schaalsymmetrie de groep van operatoren die toegestaan zijn sterk inperkt. Dit vermindert onze vrijheid zodanig dat alleen het aantal en het type van de vrijheidsgraden gekozen kan worden. Dit is een gewenst resultaat aangezien maar een klein aantal parameters invloed hebben op de voorspellingen van deze theorie (drie in het minimale geval).

Een belangrijk eigenschap van spontane symmetriebreking is de aanwezigheid van Goldstone modes, massaloze deeltjes die zich in het lage energie spectrum manifesteren. In andere woorden, Goldstone modes zijn excitaties in de richting van de gebroken symmetrie, waardoor ze massaloos zijn en in de lage-energie effectieve theorie overleven. In het geval van schaalsymmetriebreking wordt een massaloos scalair veld gecreert, vergelijkbaar met de pionen die voortkomen uit chirale symmetriebreking in QCD. In contrast tot de pionen is dit nieuwe veld neutraal onder alle Standaard Model ladingen. Vanwege de schaal-symmetrie is het scalaire veld universeel gekkopeld aan alle materievelden en kan dus waargenomen worden door middel van bestaande technieken. In Hoofdstuk 3 beschrijf ik hoe waarnemingen van zwaartekrachtgolven gebruikt kunnen worden. De zwakke koppeling met alle Standaard Model deeltjes heeft ertoe geleid dat het onzichtbaar is gebleven in alle deeltjes experimenten tot nu toe. Daar komt bij dat het alleen met scalaire velden wisselwerkt op het boom niveau, en niet meedoet aan processen met fermioon- en ijk-deeltjes. Het feit dat extreme omstandigheden kan leiden tot de creatie van deze modes, bijvoorbeeld in het bijzijn van een zwart gat of een neutronenster, maakt het waarnemen mogelijk door toekomstige zwaartekrachtgolf experimenten in de ruimte, zoals LISA.

In het geval dat het nieuwe deeltje inderdaad massaloos blijkt te zijn, dan is dit direct bewijs dat schaal-symmetrie spontaan gebroken wordt. Anderzijds, in het geval van een waargenomen massa, dan is er sprake van het bestaan van een fundamentele lengteschaal in onze theorie, die expliciet de schaal-symmetrie breekt. Dit beschrijf ik in details in Hoofdstuk 4, waar ik laat zien dat een massa kloof in de zelf-energie van Goldstone modes wordt gegenereert door expliciete symmetriebreking. In Hoofdstuk 4 wordt deze breking veroorzaakt door een constante massa parameter. Dergelijke informatie zal de natuurkunde op kleine schalen sterk inperken. Deze inperking kan mogelijk, door middel van een effectieve veldtheorie, in een modelonafhankelijke wijze gedaan worden zoals beschreven in Hoofdstuk 3.

Als laatste heb ik niet-triviale relaties tussen de verschillende scalaire velden in de theorie, zowel materievelden als zwaartekracht velden, afgeleidt. Een belangrijke rol is weggelegd voor de wijze waarmee de scalaire velden zich mengen met de zwaartekracht velden. In Hoofdstuk 1 beschrijf ik het klassieke geval van dit effect en in Hoofdstuk 4 laat ik zien hoe kwantum effecten het resultaat beinvloeden. We laten zien dat deze observabelen een niet-triviale relatie hebben met de hoekgetrouwe symmetrie. We beginnen dit nu pas te begrijpen en meer onderzoek is zeker nodig om het volledig theoretisch plaatje te krijgen.

Chapter 9

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About the author

Stefano Lucat was born in Aosta, the 28 February 1990, from Piero Lucat and Manuela Castiglion. He grew up playing and hiking in the mountains surrounding the city, developing a deep love for the mountains and all the activities that surround them. In 1996 he started playing cello, a passion that he has never abondoned since.

In 2009, he graduated from the local high school, the "Liceo scientifico É. Bèrard" and, in the same year, he also finished the first year of the bachelor of cello, at the Turin conservatorium.

Afterwards, he studied physics in Turin university, completing one year of Erasmus project in the Swedish city of Uppsala. He obtained a BsC in 2012. The same year he started a master in theoretical physics at the Dutch university of Utrecht, where he graduated *cum laude* in 2015, with a thesis with title "Cosmological singularities and bounce in Einstein-Cartan gravity".

In the fall of 2015, he started a PhD in theoretical physics, obtaining a position in the NWO graduate program of Utrecht University. This thesis is based on the work that followed from this project.

During his permanence in the Netherlands, Stefano cultivated his passion of playing cello. He played for 4 years with the Utrecht Student Concert, partaking in two shows in the concertgebouw, and joined the Netherlands Student Kamer Orchestra. In his spare time, he enjoys climbing and hiking.

Bibliography

[1] Sidney Coleman and Erick Weinberg. "Radiative Corrections as the Origin of Spontaneous Symmetry Breaking". In: *Physical Review D* 7.6 (1973), pp. 1888–1910. DOI: 10.1103/physrevd.7.1888.

[2] E. R. Harrison. "Fluctuations at the Threshold of Classical Cosmology". In: *Physical Review D* 1.10 (1970), pp. 2726–2730. DOI: 10.1103/physrevd.1.2726.

[3] Ya. B. Zeldovich. "A Hypothesis, unifying the structure and the entropy of the universe". In: *Mon. Not. Roy. Astron. Soc.* 160 (1972), 1P–3P.

[4] V.A Rubakov. "Harrison-Zeldovich spectrum from conformal invariance". In: *Journal of Cosmology and Astroparticle Physics* 2009.09 (2009), pp. 030–030. DOI: 10.1088/1475-7516/2009/09/030.

[5] E. Hubble. "A relation between distance and radial velocity among extra-galactic nebulae". In: *Proceedings of the National Academy of Sciences* 15.3 (1929), pp. 168–173. DOI: 10.1073/pnas.15.3.168.

[6] D. N. Spergel et al. "First-YearWilkinson Microwave Anisotropy Probe(WMAP) Observations: Determination of Cosmological Parameters". In: *The Astrophysical Journal Supplement Series* 148.1 (2003), pp. 175–194. DOI: 10.1086/377226.

[7] N. Aghanim et al. "Planck 2018 results. VI. Cosmological parameters". In: (2018). arXiv: 1807.06209 [astro-ph.CO].

[8] Guillem Domènech and Misao Sasaki. "Conformal frames in cosmology". In: *2nd LeCosPA Symposium: Everything about Gravity, Celebrating the Centenary of Einstein's General Relativity (LeCosPA2015) Taipei, Taiwan, December 14-18, 2015*. 2016. arXiv: 1602.06332 [gr-qc]. URL: https://inspirehep.net/record/1422742/files/arXiv:1602.06332.pdf.

[9] C. Wetterich. "Eternal Universe". In: *Phys. Rev.* D90.4 (2014), p. 043520. DOI: 10.1103/PhysRevD.90.043520. arXiv: 1404.0535 [gr-qc].

[10] Gerard T. Hooft. "Local Conformal Symmetry: the Missing Symmetry Component for Space and Time". In: (2014). arXiv: 1410.6675 [gr-qc].

[11] Krzysztof A. Meissner and Hermann Nicolai. "Conformal Symmetry and the Standard Model". In: *Phys. Lett.* B648 (2007), pp. 312–317. DOI: 10.1016/j.physletb.2007.03.023. arXiv: hep-th/0612165 [hep-th].

[12] Leonardo Chataignier et al. "Systematic analysis of radiative symmetry breaking in models with extended scalar sector". In: *Journal of High Energy Physics* 2018.8 (2018). DOI: 10.1007/jhep08(2018)083.

[13] Jean Zinn-Justin. *Quantum Field Theory and Critical Phenomena; 4th ed.* Internat. Ser. Mono. Phys. Oxford: Clarendon Press, 2002. URL: https://cds.cern.ch/record/572813.

[14] Steven Weinberg. "Critical Phenomena for Field Theorists". In: *Understanding the Fundamental Constituents of Matter*. Springer US, 1978, pp. 1–52. DOI: 10.1007/978-1-4684-0931-4_1.

[15] M. Reuter and F. Saueressig. "Renormalization group flow of quantum gravity in the Einstein-Hilbert truncation". In: *Physical Review D* 65.6 (2002). DOI: 10.1103/physrevd.65.065016.

[16] Gerard 't Hooft and M. J. G. Veltman. "One loop divergencies in the theory of gravitation". In: *Ann. Inst. H. Poincare Phys. Theor.* A20 (1974), pp. 69–94.

[17] N.D. Birrell and P.C.W. Davies. *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1984. ISBN: 9780521278584. URL: https://books.google.nl/books?id=SEnaUnrqzrUC.

[18] I. L. Shapiro. "Physical aspects of the space-time torsion". In: *Phys. Rept.* 357 (2002), p. 113. DOI: 10.1016/S0370-1573(01)00030-8. arXiv: hep-th/0103093 [hep-th].

[19] Richard P. Woodard. "Ostrogradsky's theorem on Hamiltonian instability". In: *Scholarpedia* 10.8 (2015), p. 32243. DOI: 10.4249/scholarpedia.32243. arXiv: 1506.02210 [hep-th].

[20] Gerard 't Hooft. "The Conformal Constraint in Canonical Quantum Gravity". In: (2010). arXiv: 1011.0061 [gr-qc].

[21] Michael E. Peskin, Daniel V. Schroeder, and Emil Martinec. "An Introduction to Quantum Field Theory". In: *Physics Today* 49.8 (1996), pp. 69–72. DOI: 10.1063/1.2807734.

[22] Peter Brockway Arnold. "An Introduction to baryon violation in standard electroweak theory". In: *Testing the Standard Model - TASI-90: Theoretical Advanced Study Inst. in Elementary Particle Physics Boulder, Colorado, June 3-29, 1990*. 1990, pp. 0719–742.

[23] D. M. Capper and M. J. Duff. "Trace anomalies in dimensional regularization". In: *Il Nuovo Cimento A* 23.1 (1974), pp. 173–183. DOI: 10.1007/bf02748300.

[24] Ronald J. Riegert. "A non-local action for the trace anomaly". In: *Physics Letters B* 134.1-2 (1984), pp. 56–60. DOI: 10.1016/0370-2693(84)90983-3.

[25] Maurizio Giannotti and Emil Mottola. "Trace anomaly and massless scalar degrees of freedom in gravity". In: *Physical Review D* 79.4 (2009). DOI: 10.1103/physrevd.79.045014.

[26] Nima Arkani-Hamed and Juan Maldacena. "Cosmological Collider Physics". In: (2015). arXiv: 1503.08043 [hep-th].

[27] Nima Arkani-Hamed et al. "The Cosmological Bootstrap: Inflationary Correlators from Symmetries and Singularities". In: (2018). arXiv: 1811.00024 [hep-th].

[28] Renata Kallosh and Andrei Linde. "Universality class in conformal inflation". In: *Journal of Cosmology and Astroparticle Physics* 2013.07 (2013), pp. 002–002. DOI: 10.1088/1475-7516/2013/07/002.

[29] Mehmet Ozkan and Diederik Roest. "Universality Classes of Scale Invariant Inflation". In: (2015). arXiv: 1507.03603 [hep-th].

[30] Cumrun Vafa. "The String landscape and the swampland". In: (2005). arXiv: hep-th/0509212 [hep-th].

[31] M. J. Duff. "Twenty years of the Weyl anomaly". In: *Class. Quant. Grav.* 11 (1994), pp. 1387–1404. DOI: 10.1088/0264-9381/11/6/004. arXiv: hep-th/9308075 [hep-th].

[32] Tomislav Prokopec and Jan Weenink. "Frame independent cosmological perturbations". In: *JCAP* 1309 (2013), p. 027. DOI: 10.1088/1475-7516/2013/09/027. arXiv: 1304.6737 [gr-qc].

[33] Tomislav Prokopec and Jan Weenink. "Uniqueness of the gauge invariant action for cosmological perturbations". In: *JCAP* 1212 (2012), p. 031. DOI: 10.1088/1475-7516/2012/12/031. arXiv: 1209.1701 [gr-qc].

[34] Jan Weenink and Tomislav Prokopec. "Gauge invariant cosmological perturbations for the nonminimally coupled inflaton field". In: *Phys. Rev.* D82 (2010), p. 123510. DOI: 10.1103/PhysRevD.82.123510. arXiv: 1007.2133 [hep-th].

[35] Philip D. Mannheim. "Conformal Gravity Challenges String Theory". In: *Proceedings on 13th International Symposium on Particles, strings, and cosmology (PASCOS 2007): London, UK, July 2-7, 2007*. 2007. arXiv: 0707.2283 [hep-th].

[36] Friedrich W. Hehl et al. "General relativity with spin and torsion: Foundations and prospects". In: *Rev. Mod. Phys.* 48 (3 1976), pp. 393–416. DOI: 10.1103/RevModPhys.48.393. URL: http://link.aps.org/doi/10.1103/RevModPhys.48.393.

[37] Jose W. Maluf. "Conformal Invariance and Torsion in General Relativity". In: *Gen. Rel. Grav.* 19 (1987), p. 57. DOI: 10.1007/BF01119811.

[38] J. B. Fonseca-Neto, C. Romero, and S. P. G. Martinez. "Scalar torsion and a new symmetry of general relativity". In: *General Relativity and Gravitation* 45.8 (2013), pp. 1579–1601. ISSN: 1572-9532. DOI: 10.1007/s10714-013-1553-6. URL: http://dx.doi.org/10.1007/s10714-013-1553-6.

[39] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2011. ISBN: 9780521200165, 9780521099066, 9780511826306, 9780521099066. DOI: 10.1017/CBO9780511524646.

[40] Theodore Frankel. *The Geometry of Physics*. Cambridge University Press, 2009. DOI: 10.1017/cbo9781139061377.

[41] Juan Garcia-Bellido et al. "Higgs-Dilaton Cosmology: From the Early to the Late Universe". In: *Phys. Rev.* D84 (2011), p. 123504. DOI: 10.1103/PhysRevD.84.123504. arXiv: 1107.2163 [hep-ph].

[42] Stefano Lucat and Tomislav Prokopec. "Cosmological singularities and bounce in Cartan-Einstein theory". In: *Journal of Cosmology and Astroparticle Physics* 2017.10 (2017), pp. 047–047. DOI: 10.1088/1475-7516/2017/10/047.

[43] Dražen Glavan and Tomislav Prokopec. "Nonminimal coupling and the cosmological constant problem". In: (2015). arXiv: 1504.00842 [gr-qc].

[44] H. T. Nieh and M. L. Yan. "An identity in Riemann–Cartan geometry". In: *Journal of Mathematical Physics* 23.3 (1982), pp. 373–374. DOI: 10.1063/1.525379. eprint: https://doi.org/10.1063/1.525379. URL: https://doi.org/10.1063/1.525379.

[45] A.A. Starobinsky. "A new type of isotropic cosmological models without singularity". In: *Physics Letters B* 91.1 (1980), pp. 99–102. DOI: 10.1016/0370-2693(80)90670-x.

[46] Alan H. Guth. "Inflationary universe: A possible solution to the horizon and flatness problems". In: *Physical Review D* 23.2 (1981), pp. 347–356. DOI: 10.1103/physrevd.23.347.

[47] Jérôme Martin, Christophe Ringeval, and Vincent Vennin. "Encyclopædia Inflationaris". In: *Physics of the Dark Universe* 5-6 (2014), pp. 75–235. DOI: 10.1016/j.dark.2014.01.003.

[48] Clifford Cheung et al. "The effective field theory of inflation". In: *Journal of High Energy Physics* 2008.03 (2008), pp. 014–014. DOI: 10.1088/1126-6708/2008/03/014.

[49] C. Wetterich. "Cosmology and the fate of dilatation symmetry". In: *Nuclear Physics B* 302.4 (1988), pp. 668–696. DOI: 10.1016/0550-3213(88)90193-9.

[50] Fedor Bezrukov, Javier Rubio, and Mikhail Shaposhnikov. "Living beyond the edge: Higgs inflation and vacuum metastability". In: *Physical Review D* 92.8 (2015). DOI: 10.1103/physrevd.92.083512.

[51] Georgios K. Karananas and Mikhail Shaposhnikov. "Scale-invariant alternatives to general relativity. II. Dilaton properties". In: *Physical Review D* 93.8 (2016). DOI: 10.1103/physrevd.93.084052.

[52] Renata Kallosh and Andrei Linde. "Non-minimal Inflationary Attractors". In: *Journal of Cosmology and Astroparticle Physics* 2013.10 (2013), pp. 033–033. DOI: 10.1088/1475-7516/2013/10/033.

[53] John Joseph M. Carrasco, Renata Kallosh, and Andrei Linde. "Cosmological attractors and initial conditions for inflation". In: *Physical Review D* 92.6 (2015). DOI: `10.1103/physrevd.92.063519`.

[54] Pedro G. Ferreira, Christopher T. Hill, and Graham G. Ross. "Scale-independent inflation and hierarchy generation". In: *Physics Letters B* 763 (2016), pp. 174–178. DOI: `10.1016/j.physletb.2016.10.036`.

[55] Pedro G. Ferreira, Christopher T. Hill, and Graham G. Ross. "Weyl current, scale-invariant inflation, and Planck scale generation". In: *Physical Review D* 95.4 (2017). DOI: `10.1103/physrevd.95.043507`.

[56] Pedro G. Ferreira et al. "Inflation in a scale-invariant universe". In: *Physical Review D* 97.12 (2018). DOI: `10.1103/physrevd.97.123516`.

[57] Javier Rubio and Christof Wetterich. "Emergent scale symmetry: Connecting inflation and dark energy". In: *Physical Review D* 96.6 (2017). DOI: `10.1103/physrevd.96.063509`.

[58] K. Kannike et al. "The evolving Planck mass in classically scale-invariant theories". In: *Journal of High Energy Physics* 2017.4 (2017). DOI: `10.1007/jhep04(2017)026`.

[59] Alberto Salvio and Alessandro Strumia. "Agravity up to infinite energy". In: *The European Physical Journal C* 78.2 (2018). DOI: `10.1140/epjc/s10052-018-5588-4`.

[60] Alberto Salvio. "Inflationary perturbations in no-scale theories". In: *The European Physical Journal C* 77.4 (2017). DOI: `10.1140/epjc/s10052-017-4825-6`.

[61] Alberto Salvio and Alessandro Strumia. "Agravity". In: *Journal of High Energy Physics* 2014.6 (2014). DOI: `10.1007/jhep06(2014)080`.

[62] Massimiliano Rinaldi and Luciano Vanzo. "Inflation and reheating in theories with spontaneous scale invariance symmetry breaking". In: *Physical Review D* 94.2 (2016). DOI: `10.1103/physrevd.94.024009`.

[63] Georgios K. Karananas and Javier Rubio. "On the geometrical interpretation of scale-invariant models of inflation". In: *Physics Letters B* 761 (2016), pp. 223–228. DOI: `10.1016/j.physletb.2016.08.037`.

[64] Santiago Casas, Martin Pauly, and Javier Rubio. "Higgs-dilaton cosmology: An inflation–dark-energy connection and forecasts for future galaxy surveys". In: *Physical Review D* 97.4 (2018). DOI: `10.1103/physrevd.97.043520`.

[65] Yong Tang and Yue-Liang Wu. "Inflation in gauge theory of gravity with local scaling symmetry and quantum induced symmetry breaking". In: *Physics Letters B* 784 (2018), pp. 163–168. DOI: `10.1016/j.physletb.2018.07.048`.

[66] Ana Achúcarro et al. "Universality of multi-field α-attractors". In: *Journal of Cosmology and Astroparticle Physics* 2018.04 (2018), pp. 028–028. DOI: `10.1088/1475-7516/2018/04/028`.

[67] Remko Klein, Diederik Roest, and David Stefanyszyn. "Symmetry breaking patterns for inflation". In: *Journal of High Energy Physics* 2018.6 (2018). DOI: 10.1007/jhep06(2018)006.

[68] K. Sravan Kumar and Paulo Vargas Moniz. "Conformal GUT inflation, proton lifetime and non-thermal leptogenesis". In: (2018). arXiv: 1806.09032 [hep-ph].

[69] A. Karam et al. "Constant-roll (quasi-)linear inflation". In: *Journal of Cosmology and Astroparticle Physics* 2018.05 (2018), pp. 011–011. DOI: 10.1088/1475-7516/2018/05/011.

[70] Luca Marzola and Antonio Racioppi. "Minimal but non-minimal inflation and electroweak symmetry breaking". In: *Journal of Cosmology and Astroparticle Physics* 2016.10 (2016), pp. 010–010. DOI: 10.1088/1475-7516/2016/10/010.

[71] Kristjan Kannike, Antonio Racioppi, and Martti Raidal. "Linear inflation from quartic potential". In: *Journal of High Energy Physics* 2016.1 (2016). DOI: 10.1007/jhep01(2016)035.

[72] Stefano Lucat and Tomislav Prokopec. "The role of conformal symmetry in gravity and the standard model". In: *Classical and Quantum Gravity* 33.24 (2016), p. 245002. DOI: 10.1088/0264-9381/33/24/245002.

[73] Sébastien Renaux-Petel and Krzysztof Turzyński. "Geometrical Destabilization of Inflation". In: *Physical Review Letters* 117.14 (2016). DOI: 10.1103/physrevlett.117.141301.

[74] Stefano Lucat, Tomislav Prokopec, and Bogumiła Świeżewska. "Conformal symmetry and the cosmological constant problem". In: *International Journal of Modern Physics D* 27.14 (2018), p. 1847014. DOI: 10.1142/s0218271818470144.

[75] Daniel Baumann, Hayden Lee, and Guilherme L. Pimentel. "High-scale inflation and the tensor tilt". In: *Journal of High Energy Physics* 2016.1 (2016). DOI: 10.1007/jhep01(2016)101.

[76] Ana Achúcarro et al. "Mass hierarchies and nondecoupling in multi-scalar-field dynamics". In: *Physical Review D* 84.4 (2011). DOI: 10.1103/physrevd.84.043502.

[77] Boris Spokoiny. "Deflationary Universe scenario". In: *Physics Letters B* 315.1-2 (1993), pp. 40–45. DOI: 10.1016/0370-2693(93)90155-b.

[78] Michael Joyce and Tomislav Prokopec. "Turning around the sphaleron bound: Electroweak baryogenesis in an alternative post-inflationary cosmology". In: *Physical Review D* 57.10 (1998), pp. 6022–6049. DOI: 10.1103/physrevd.57.6022.

[79] Alexei A. Starobinsky. "Spectrum of adiabatic perturbations in the universe when there are singularities in the inflation potential". In: *JETP Lett.* 55 (1992). [Pisma Zh. Eksp. Teor. Fiz.55,477(1992)], pp. 489–494.

[80] D. J. Brooker, N. C. Tsamis, and R. P. Woodard. "Precision predictions for the primordial power spectra of scalar potential models of inflation". In: *Physical Review D* 93.4 (2016). DOI: 10.1103/physrevd.93.043503.

[81] C. L. Bennett et al. "Four-Year [ITAL]COBE[/ITAL] DMR Cosmic Microwave Background Observations: Maps and Basic Results". In: *The Astrophysical Journal* 464.1 (1996), pp. L1–L4. DOI: 10.1086/310075.

[82] Anja Marunović and Tomislav Prokopec. "Planck scale operators, inflation, and fine tuning". In: *Physical Review D* 96.4 (2017). DOI: 10.1103/physrevd.96.043512.

[83] A.O. Barvinsky and G.A. Vilkovisky. "The generalized Schwinger-Dewitt technique in gauge theories and quantum gravity". In: *Physics Reports* 119.1 (1985), pp. 1–74. DOI: 10.1016/0370-1573(85)90148-6.

[84] Gerard 't Hooft. "Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking". In: *NATO Sci. Ser. B* 59 (1980), pp. 135–157. DOI: 10.1007/978-1-4684-7571-5_9.

[85] Jurjen F. Koksma and Tomislav Prokopec. "The Cosmological Constant and Lorentz Invariance of the Vacuum State". In: (2011). arXiv: 1105.6296 [gr-qc].

[86] Stefano Lucat and Tomislav Prokopec. "Is conformal symmetry really anomalous?" In: (2017). arXiv: 1709.00330 [gr-qc].

[87] Stefan Nobbenhuis. "Categorizing Different Approaches to the Cosmological Constant Problem". In: *Foundations of Physics* 36.5 (2006), pp. 613–680. DOI: 10.1007/s10701-005-9042-8.

[88] Steven Weinberg. "The cosmological constant problem". In: *Reviews of Modern Physics* 61.1 (1989), pp. 1–23. DOI: 10.1103/revmodphys.61.1.

[89] P. J. E. Peebles and Bharat Ratra. "The cosmological constant and dark energy". In: *Reviews of Modern Physics* 75.2 (2003), pp. 559–606. DOI: 10.1103/revmodphys.75.559.

[90] S.E. Rugh and H. Zinkernagel. "The quantum vacuum and the cosmological constant problem". In: *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 33.4 (2002), pp. 663–705. DOI: 10.1016/s1355-2198(02)00033-3.

[91] Alexander Barnaveli, Stefano Lucat, and Tomislav Prokopec. "Inflation as a spontaneous symmetry breaking of Weyl symmetry". In: *Journal of Cosmology and Astroparticle Physics* 2019.01 (2019), pp. 022–022. DOI: 10.1088/1475-7516/2019/01/022.

[92] C. Ford et al. "The effective potential and the renormalisation group". In: *Nuclear Physics B* 395.1-2 (1993), pp. 17–34. DOI: 10.1016/0550-3213(93)90206-5.

[93] William A. Bardeen. "On naturalness in the standard model". In: *Ontake Summer Institute on Particle Physics Ontake Mountain, Japan, August 27-September 2, 1995*. 1995. URL: http://lss.fnal.gov/cgi-bin/find_paper.pl?conf-95-391.

[94] E. Di Valentino et al. "Exploring cosmic origins with CORE: Cosmological parameters". In: *Journal of Cosmology and Astroparticle Physics* 2018.04 (2018), pp. 017–017. DOI: 10.1088/1475-7516/2018/04/017.

[95] B. P. Abbott et al. "Multi-messenger Observations of a Binary Neutron Star Merger". In: *The Astrophysical Journal* 848.2 (2017), p. L12. DOI: 10.3847/2041-8213/aa91c9.

[96] Pedro G. Ferreira, Christopher T. Hill, and Graham G. Ross. "No fifth force in a scale invariant universe". In: *Physical Review D* 95.6 (2017). DOI: 10.1103/physrevd.95.064038.

[97] Nikodem Popławski. "Nonsingular, big-bounce cosmology from spinor-torsion coupling". In: *Physical Review D* 85.10 (2012). DOI: 10.1103/physrevd.85.107502.

[98] Nikodem J. Popławski. "Big bounce from spin and torsion". In: *General Relativity and Gravitation* 44.4 (2012), pp. 1007–1014. DOI: 10.1007/s10714-011-1323-2.

[99] Nikodem J. Popławski. "Erratum to "Cosmology with torsion: An alternative to cosmic inflation" [Phys. Lett. B 694 (2010) 181]". In: *Physics Letters B* 701.5 (2011), p. 672. DOI: 10.1016/j.physletb.2011.05.047.

[100] Yi-Peng Wu. "Inflation with teleparallelism: Can torsion generate primordial fluctuations without local Lorentz symmetry?" In: *Physics Letters B* 762 (2016), pp. 157–161. DOI: 10.1016/j.physletb.2016.09.025.

[101] L.C. Garcia de Andrade. "Primordial magnetic fields and dynamos from parity violated torsion". In: *Physics Letters B* 711.2 (2012), pp. 143–146. DOI: 10.1016/j.physletb.2012.03.075.

[102] Alexander S. Belyaev, Marc C. Thomas, and Ilya L. Shapiro. "Torsion as a dark matter candidate from the Higgs portal". In: *Physical Review D* 95.9 (2017). DOI: 10.1103/physrevd.95.095033.

[103] M. I. Wanas. "Dark Energy: Is It of Torsion Origin?" In: *1st Middle-East African Regional IAU Meeting: MEARIM 2008 Cairo, Egypt, April 5-10, 2008*. 2010. arXiv: 1006.2154 [gr-qc].

[104] Nikodem J. Popławski. "Four-fermion interaction from torsion as dark energy". In: *General Relativity and Gravitation* 44.2 (2011), pp. 491–499. DOI: 10.1007/s10714-011-1288-1.

[105] A. S. Belyaev, I. L. Shapiro, and M. A. B. do Vale. "Torsion phenomenology at the CERN LHC". In: *Physical Review D* 75.3 (2007). DOI: 10.1103/physrevd.75.034014.

[106] Jeffrey Goldstone, Abdus Salam, and Steven Weinberg. "Broken Symmetries". In: *Physical Review* 127.3 (1962), pp. 965–970. DOI: 10.1103/physrev.127.965.

[107] Yoshimasa Hidaka, Toshifumi Noumi, and Gary Shiu. "Effective field theory for spacetime symmetry breaking". In: *Physical Review D* 92.4 (2015). DOI: 10.1103/physrevd.92.045020.

[108] Hiren H. Patel. "Package -X: A Mathematica package for the analytic calculation of one-loop integrals". In: *Computer Physics Communications* 197 (2015), pp. 276–290. DOI: 10.1016/j.cpc.2015.08.017.

[109] Joseph Polchinski. "Scale and conformal invariance in quantum field theory". In: *Nuclear Physics B* 303.2 (1988), pp. 226–236. DOI: 10.1016/0550-3213(88)90179-4.

[110] Curtis G. Callan. "Broken Scale Invariance in Scalar Field Theory". In: *Physical Review D* 2.8 (1970), pp. 1541–1547. DOI: 10.1103/physrevd.2.1541.

[111] Curtis G Callan, Sidney Coleman, and Roman Jackiw. "A new improved energy-momentum tensor". In: *Annals of Physics* 59.1 (1970), pp. 42–73. DOI: 10.1016/0003-4916(70)90394-5.

[112] F. Englert, C. Truffin, and R. Gastmans. "Conformal invariance in quantum gravity". In: *Nuclear Physics B* 117.2 (1976), pp. 407–432. DOI: 10.1016/0550-3213(76)90406-5.

[113] D. M. Ghilencea. "Manifestly scale-invariant regularization and quantum effective operators". In: *Physical Review D* 93.10 (2016). DOI: 10.1103/physrevd.93.105006.

[114] Ignatios Antoniadis, Pawel O Mazur, and Emil Mottola. "Cosmological dark energy: prospects for a dynamical theory". In: *New Journal of Physics* 9.1 (2007), pp. 11–11. DOI: 10.1088/1367-2630/9/1/011.

[115] S. Alekhin, A. Djouadi, and S. Moch. "The top quark and Higgs boson masses and the stability of the electroweak vacuum". In: *Physics Letters B* 716.1 (2012), pp. 214–219. DOI: 10.1016/j.physletb.2012.08.024.

[116] Giuseppe Degrassi et al. "Higgs mass and vacuum stability in the Standard Model at NNLO". In: *Journal of High Energy Physics* 2012.8 (2012). DOI: 10.1007/jhep08(2012)098.

[117] Satsuki Oda, Nobuchika Okada, and Dai suke Takahashi. "Classically conformalU(1)′extended standard model and Higgs vacuum stability". In: *Physical Review D* 92.1 (2015). DOI: 10.1103/physrevd.92.015026.

[118] H. Weyl. "Eine neue Erweiterung der Relativitätstheorie". In: *Annalen der Physik* 364.10 (1919), pp. 101–133. DOI: 10.1002/andp.19193641002.